Differential Pulse Code Modulation

- The quantizers, that we studied so far, are memoryless, in the sense that quantization is done on a sample-by-sample basis. Each sample is quantized and encoded into n binary digits, regardless of any correlation with other samples.

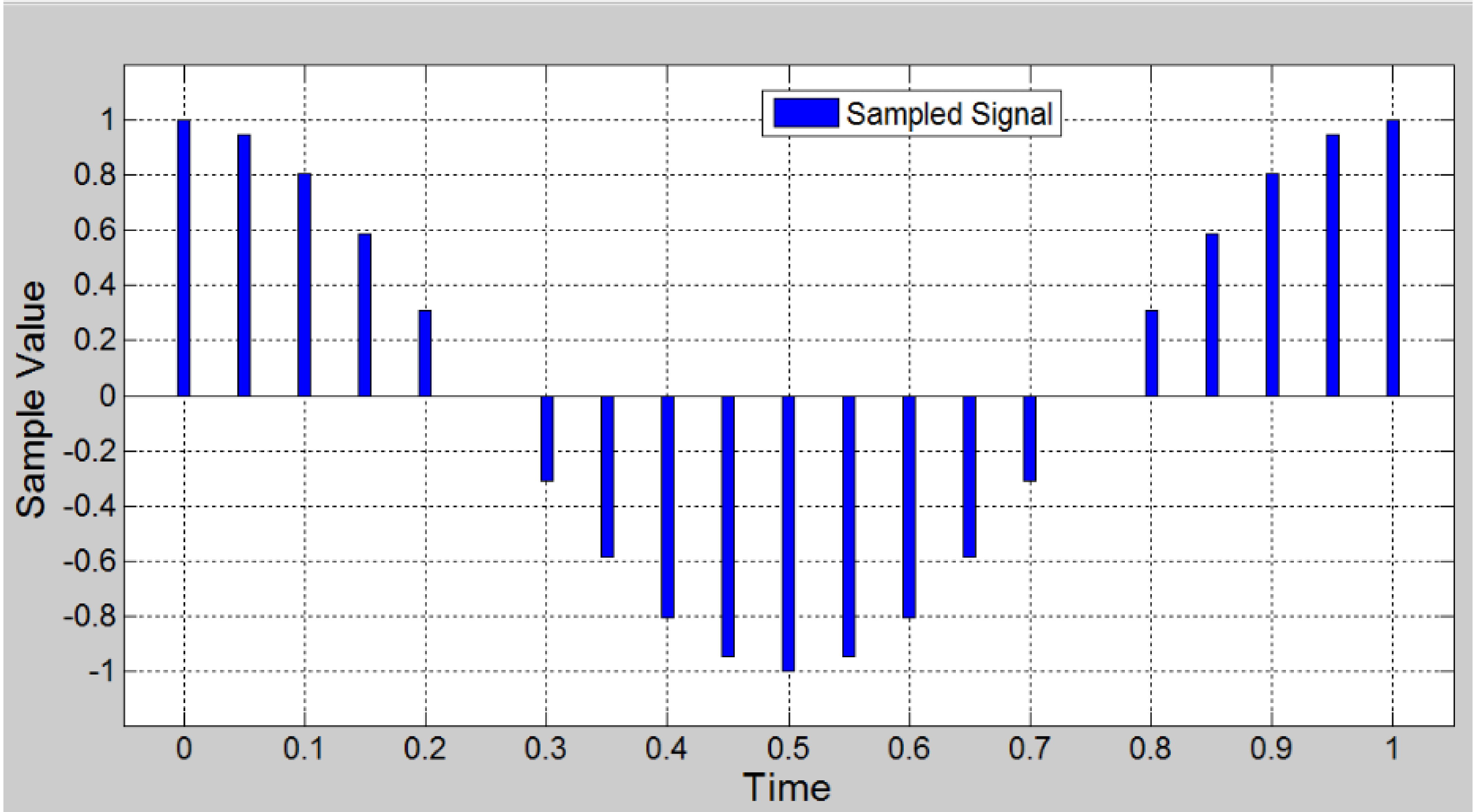
- A ***differential pulse-code modulation (DPCM)*** *quantizer* quantizes the difference between a sample and a predicted value of that sample. Here, correlation between successive samples is utilized.

- The prediction is based, in general, on past m samples of the signal. If successive samples are highly correlated, the predictor output will be very close to the next sample value, and hence the prediction error will be small.

- An error with a small variance further means that **fewer bits (r<n) are needed to represent the error**.

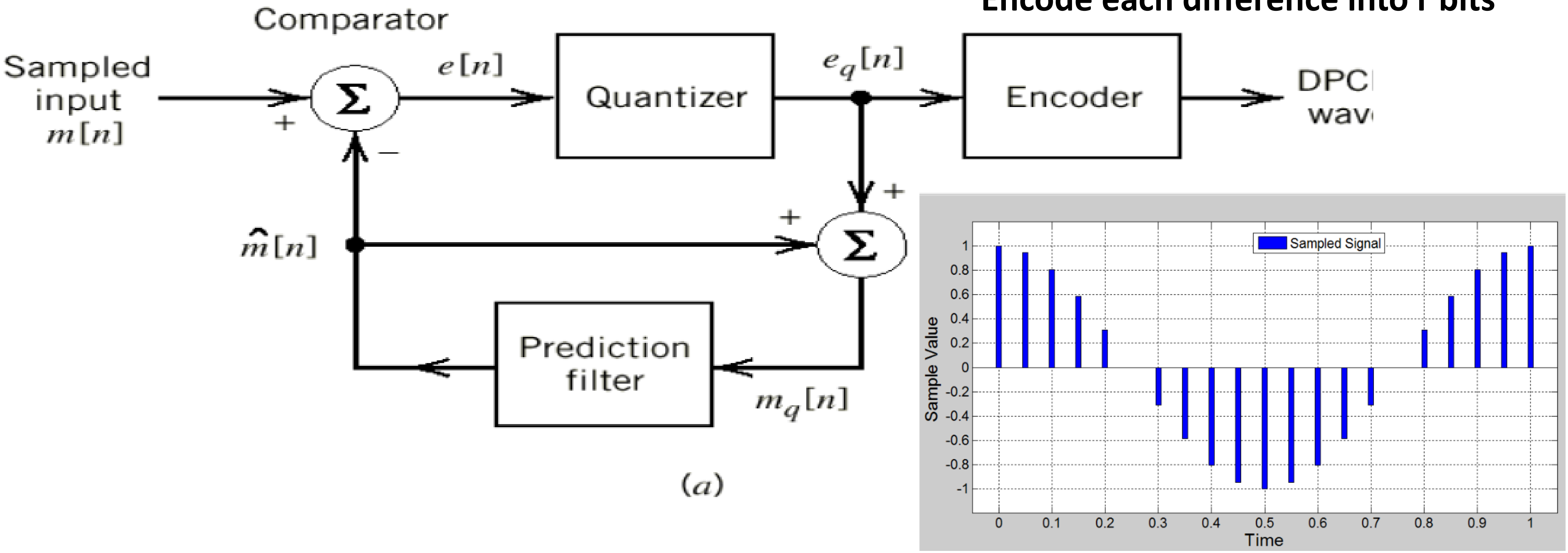
- At the receiver, a predictor similar to the one used at the transmitter is used to reconstruct the original waveform

Differential Pulse Code Modulation

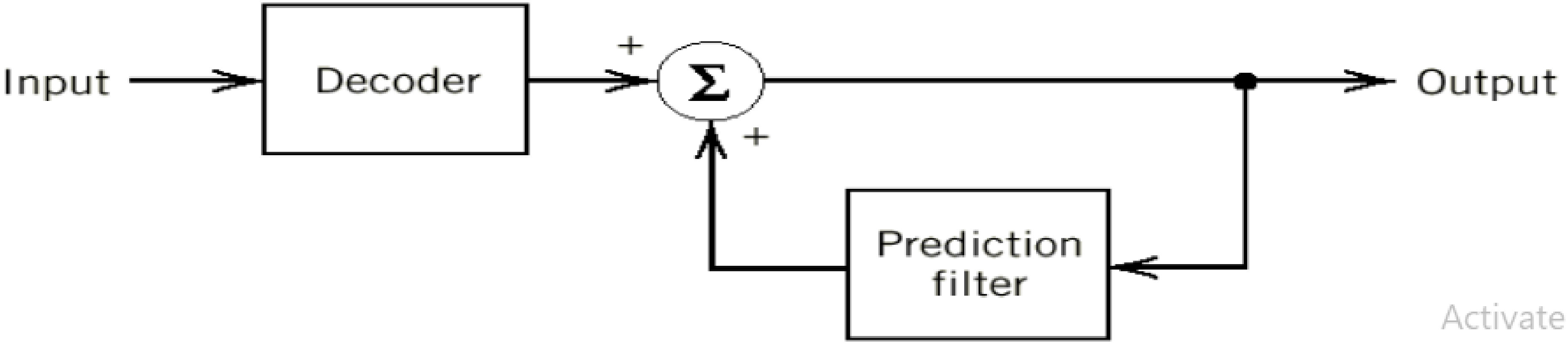


DPCM: Basic Operation

Encode each difference into r bits



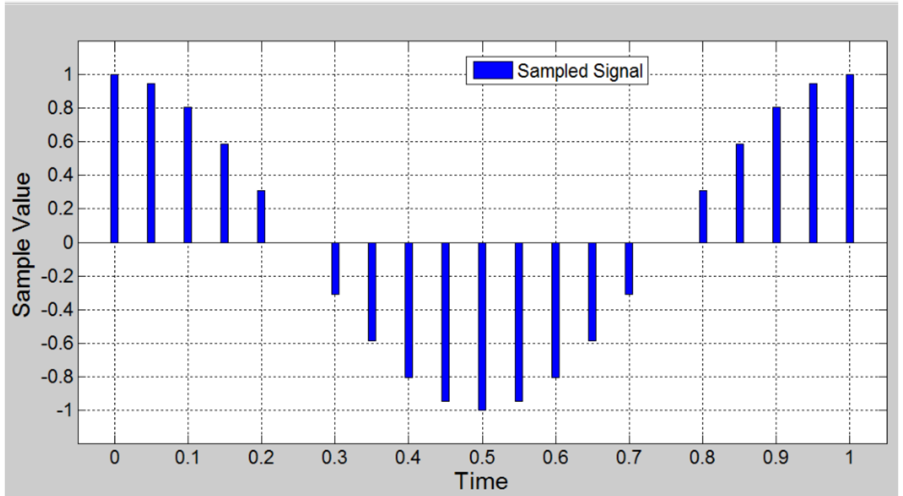
(a)



DPCM: Linear Prediction Filter

It is a discrete-time, finite-duration impulse response filter (FIR), which consists of three blocks:

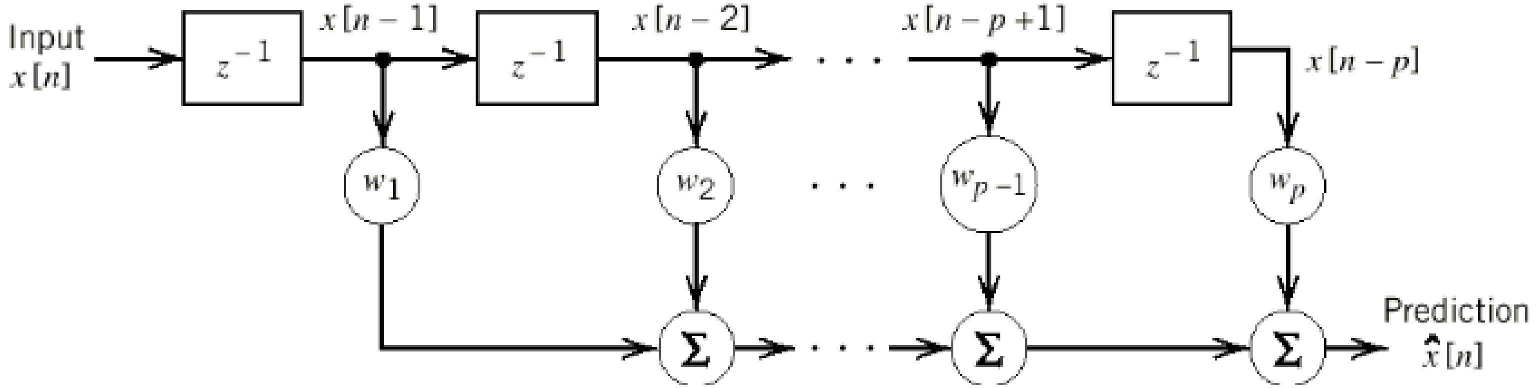
1. Set of p (p: prediction order) unit-delay elements (z^{-1})
2. Set of multipliers with coefficients $w_1, w_2, \ldots w_p$
3. Set of adders ($\sum$)



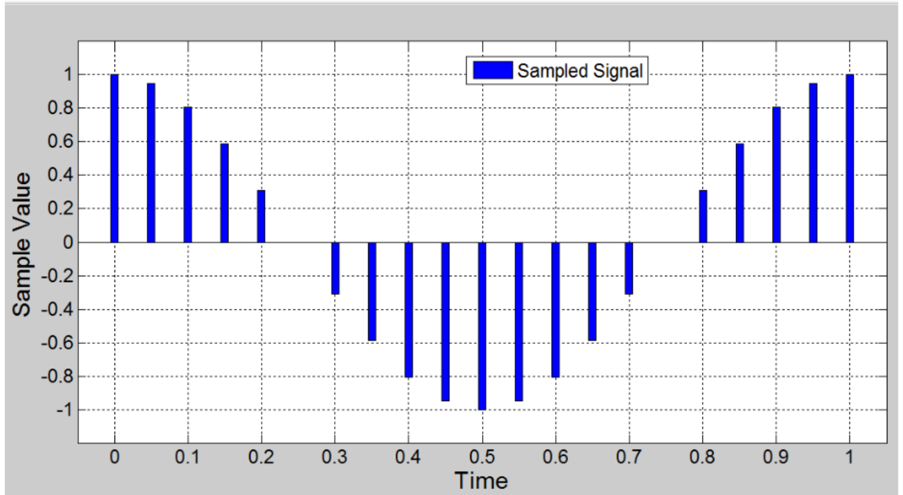
This filter expresses the predicted value of the sample at time (nT_s) as a linear combination of the past p samples of the signal.

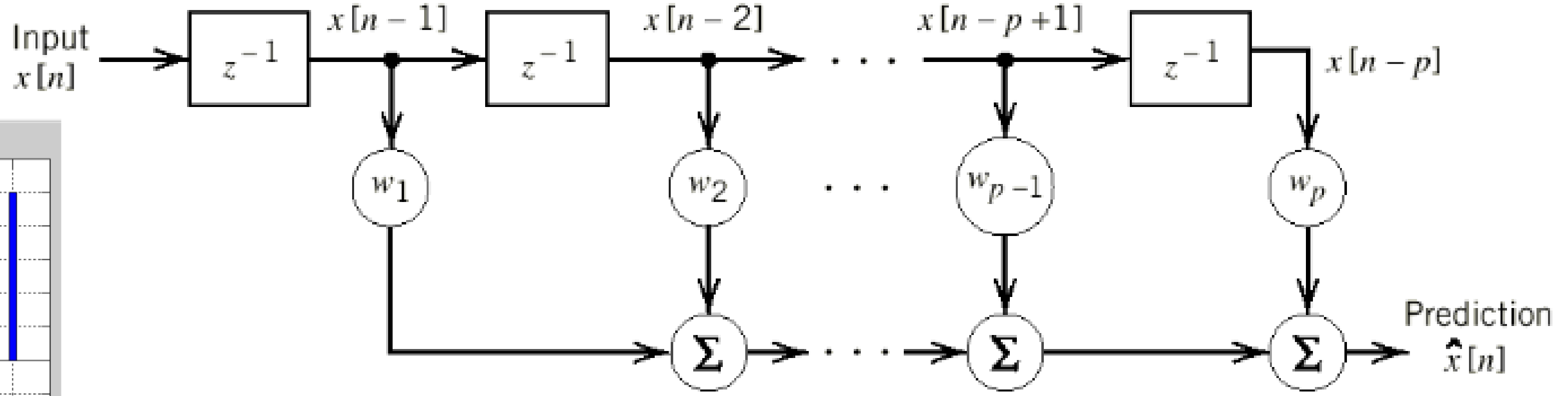
$$\hat{x}(n) = w_1 x(n-1) + w_2 x(n-2) + \cdots + w_p x(n-p)$$

The coefficients w_1, w_2, …, w_p are chosen so as to minimize the mean square error $E(x(n) - \hat{x}(n))^2$.



DPCM: Linear Prediction Filter





$$\epsilon = E((x(n) - \hat{x}(n))^2; \qquad \text{prediction error}$$

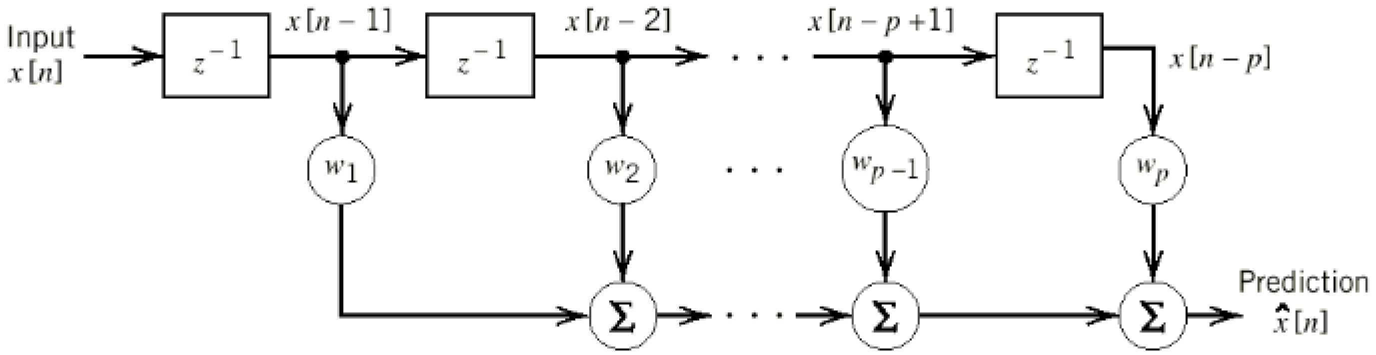
Substituting $\hat{x}(n) = w_1 x(n-1) + w_2 x(n-2) + \cdots + w_p x(n-p)$, the prediction error becomes:

$$\epsilon = E((x(n) - w_1 x(n-1) + w_2 x(n-2) + \cdots + w_p x(n-p))^2$$

Expanding ϵ and taking expectation of all terms, we get:

$$\epsilon = E(x(n)^2) - 2\sum_{i=1}^{p} w_i E[x(n)x(n-i)] + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \; E[x(n-i)x(n-j)]$$

DPCM: Linear Prediction Filter



Recognize that: $\mathbf{R_x(i) = E[x(n)x(n-i)]}$ is the autocorrelation function of $x(t)$.

Differentiating ϵ with respect to w_i, setting the derivative to zero, and solving, we get (assuming p=3)

$$\begin{bmatrix} R_x(0) & R_x(1) & R_x(2) \\ R_x(-1) & R_x(0) & R_x(1) \\ R_x(-2) & R_x(-1) & R_x(0) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} R_x(1) \\ R_x(2) \\ R_x(3) \end{bmatrix}$$

$$R(1) = R(T_s)$$
$$R(2) = R(2T_s)$$

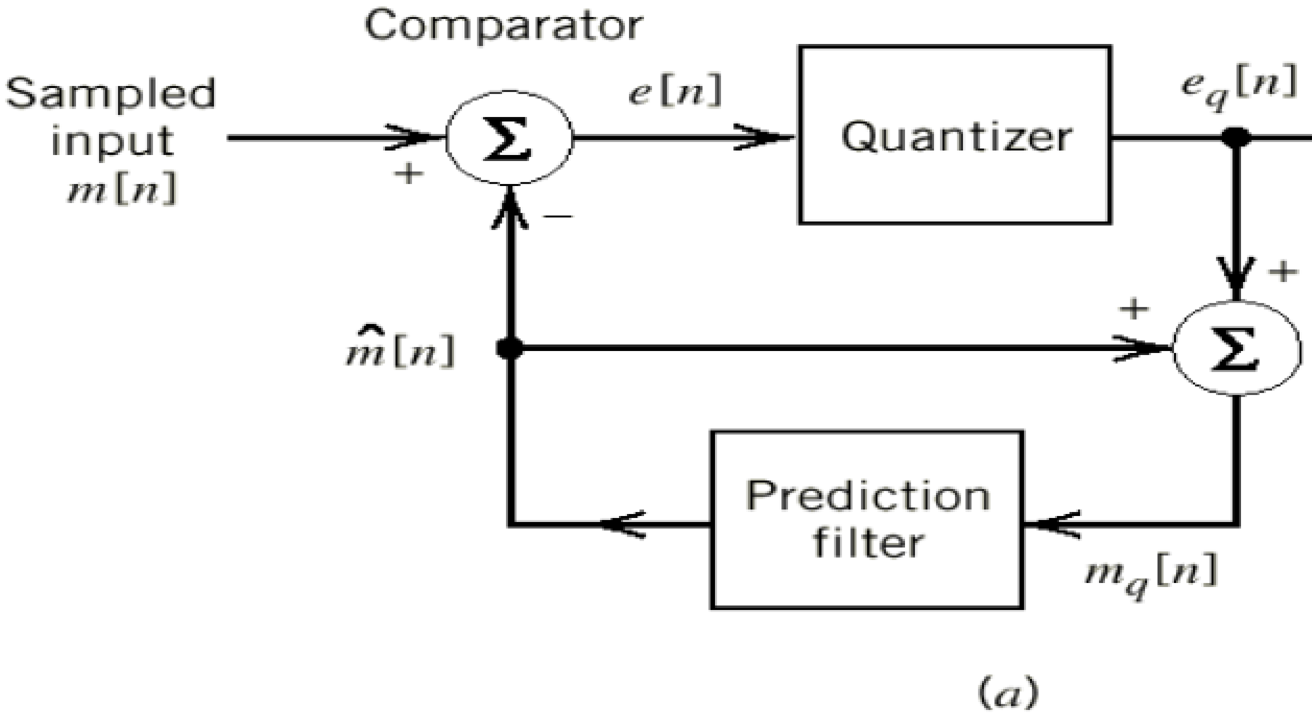
If p=1, the above equation reduces to

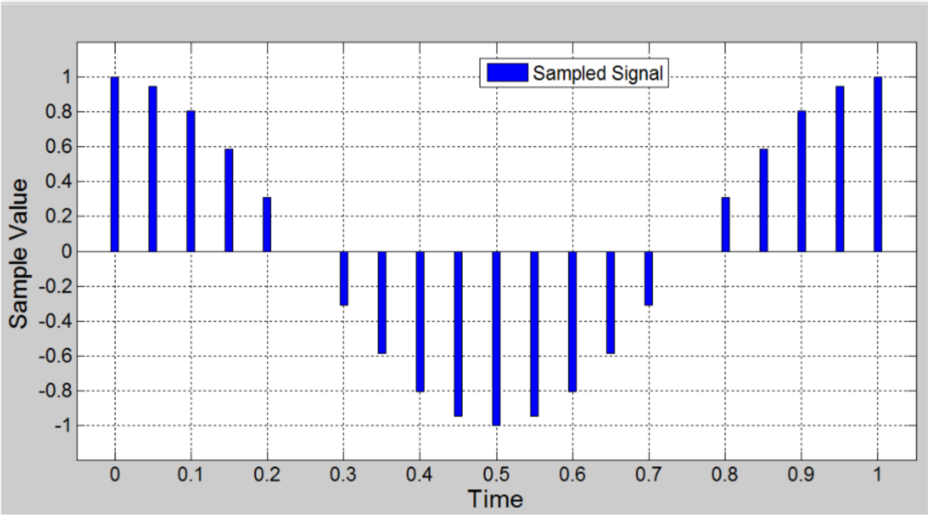
$$w_1 = R_x(1)/\, R_x(0)$$

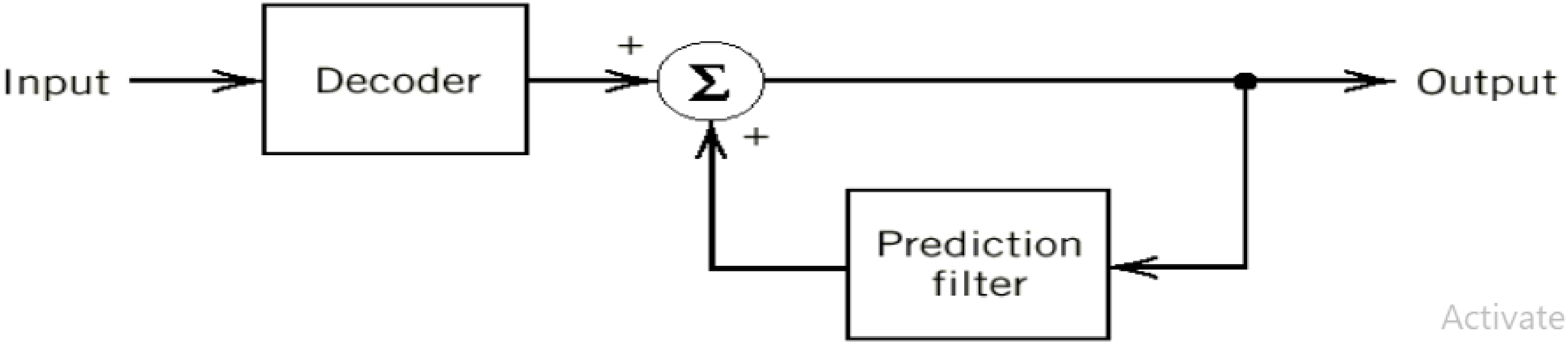
Note that: $R_x(-1) = R_x(1), R_x(-2) = R_x(2), R_x(1) = R_x(T_s), R_x(2) = R_x(2T_s), R_x(3) = R_x(3T_s)$.

DPDM: Transmitter and Receiver

Encode each difference into r bits



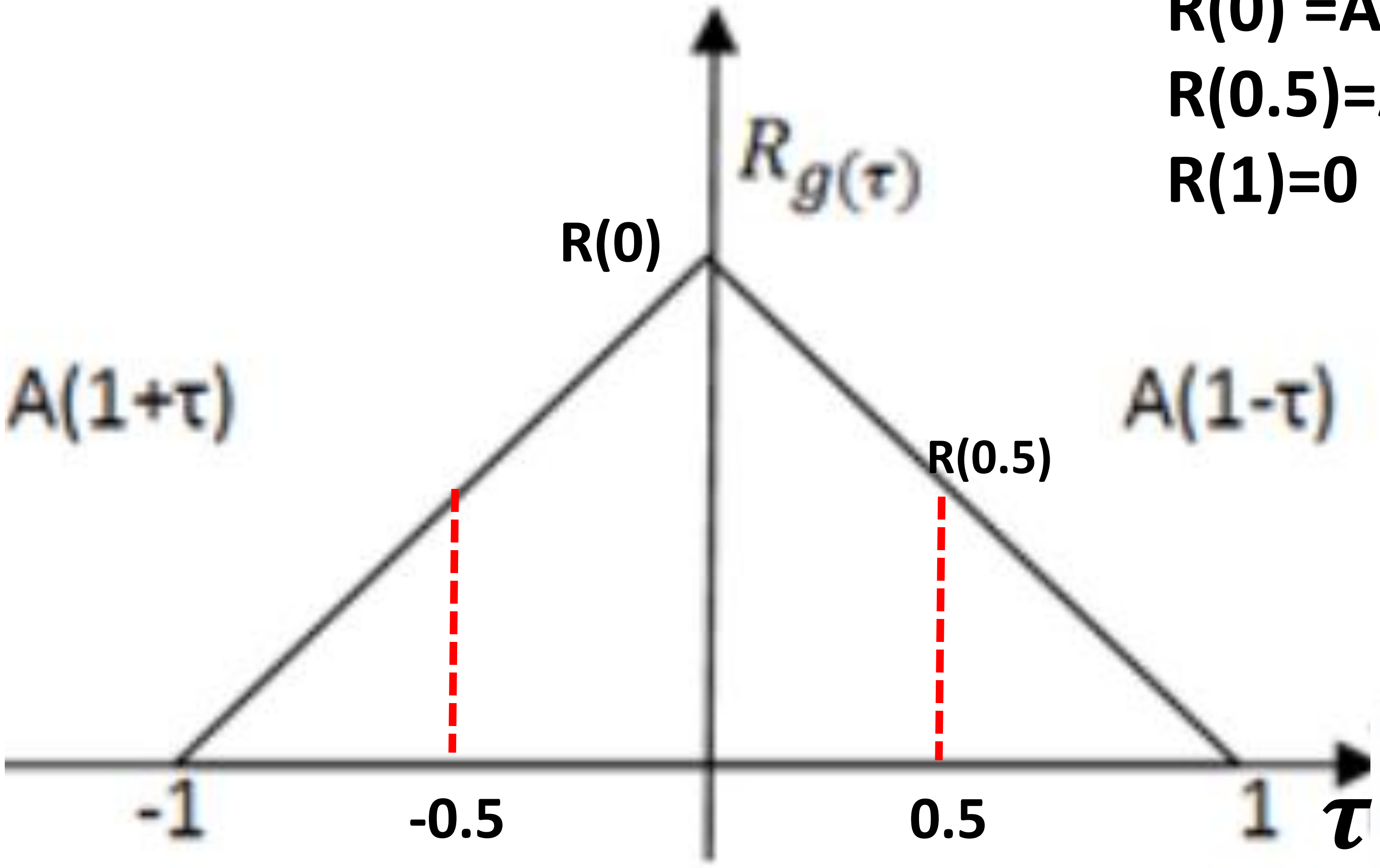




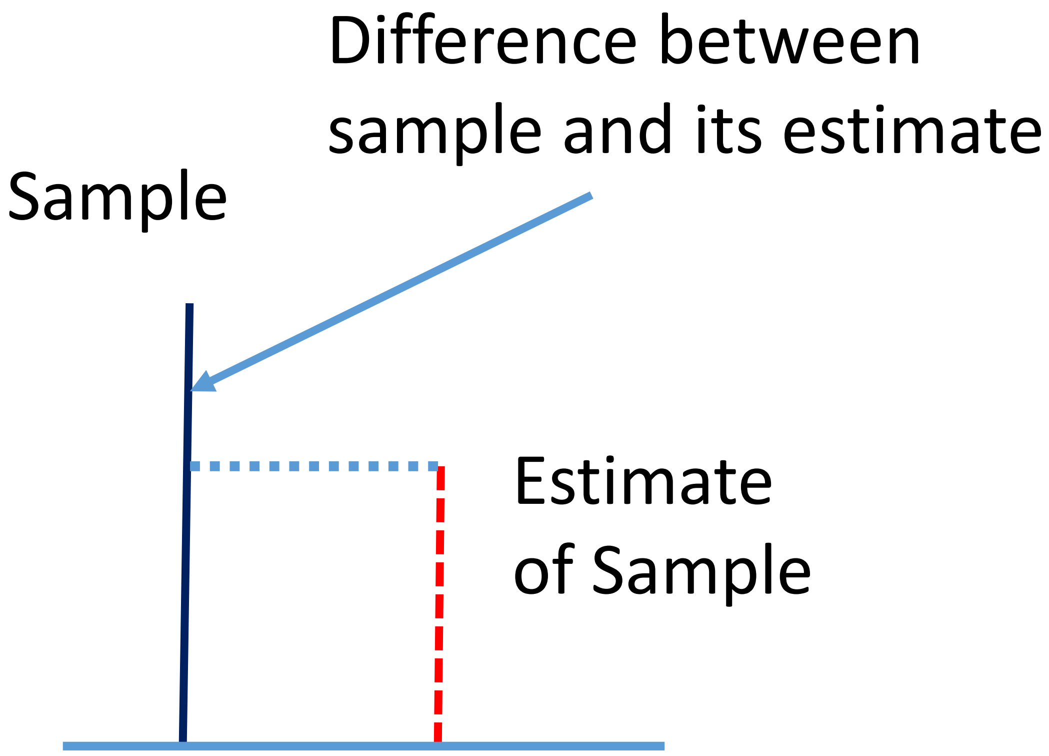
DPCM: Autocorrelation Function

R is an even function

R(0) =A
R(0.5)=A/2
R(1)=0



DPCM: Concluding Summary

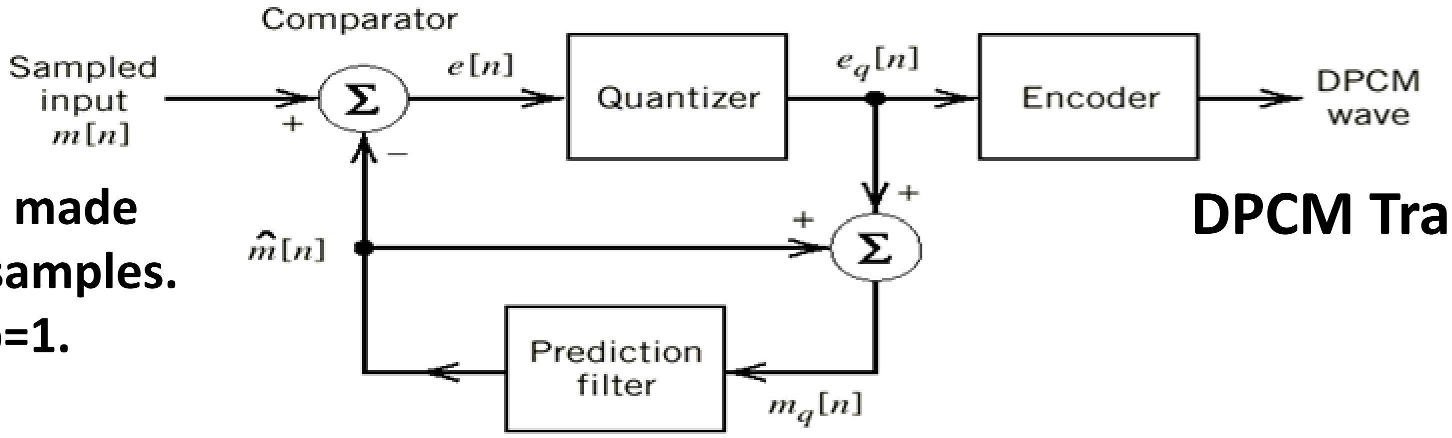


- **At transmitter**:
  - Samples are known
  - Estimate is known since estimate is a linear function of the samples.
- Transmit
- Difference= Sample – Estimate
- **At receiver**:
  - Receive Difference
  - Construct Estimate
- Sample = Estimate + Difference

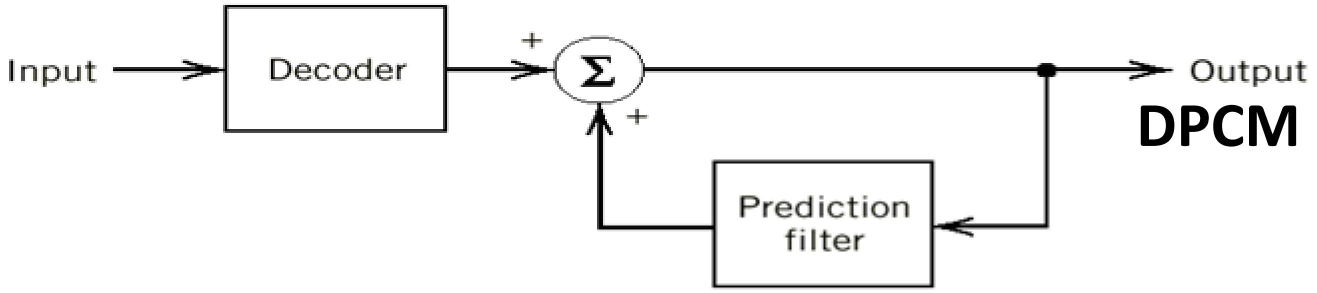
Delta Modulation

- **Remark:** Before you attend this lecture, please attend the previous one on DPCM.
- Delta modulation (DM), is a special case of Differential Pulse Code Modulation (DPCM).
- The order of the prediction filter in delta modulation is **p=1** and **represents only the quantized value of the previous sample**. The **number of quantization levels is two.**
- In this scheme, the system transmits the sign of the difference between the current sample and the quantized value of the previous sample. The **sign is represented by a single bit**.

Prediction in DPCM is made based on p previous samples. In delta modulation p=1.

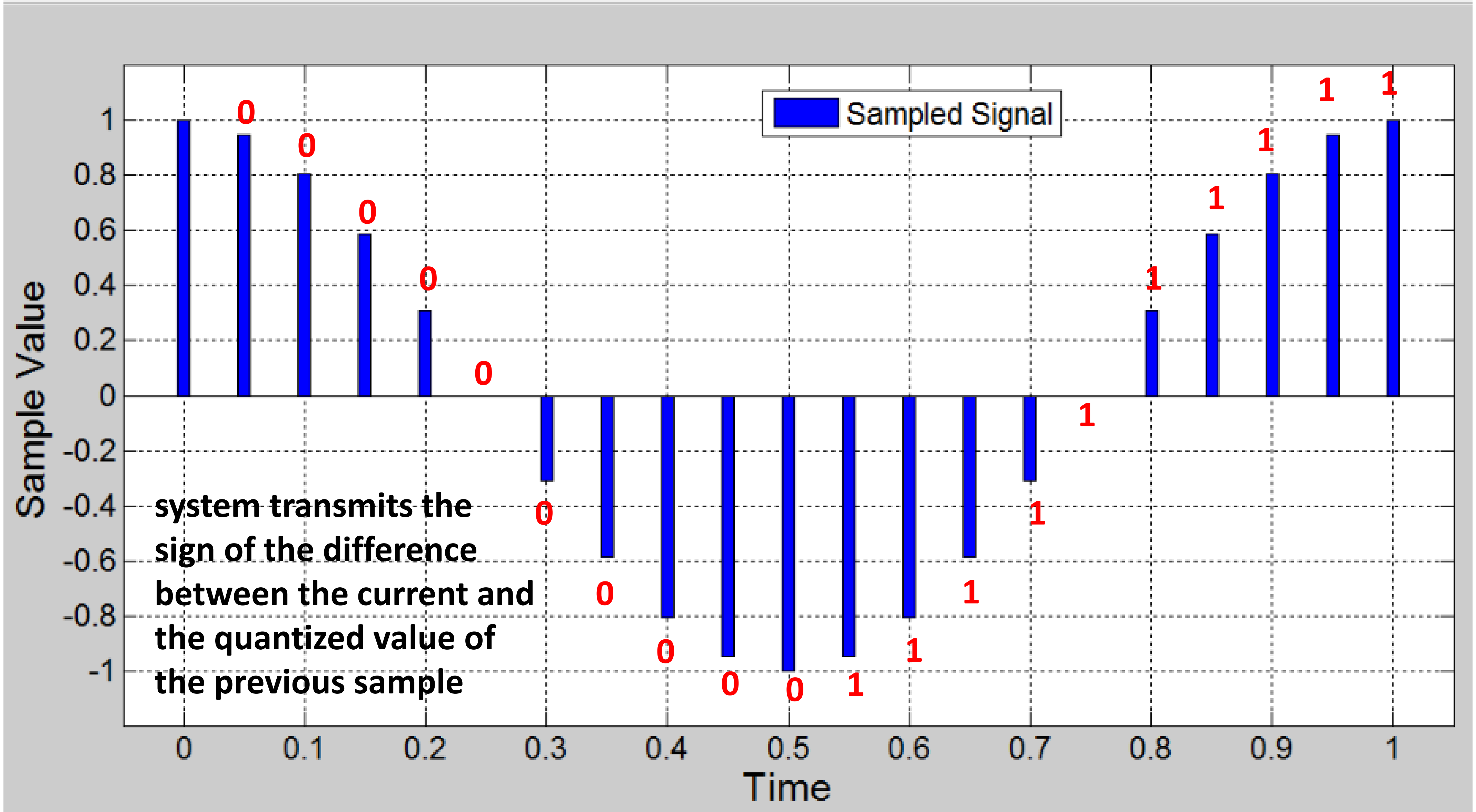


DPCM Transmitter



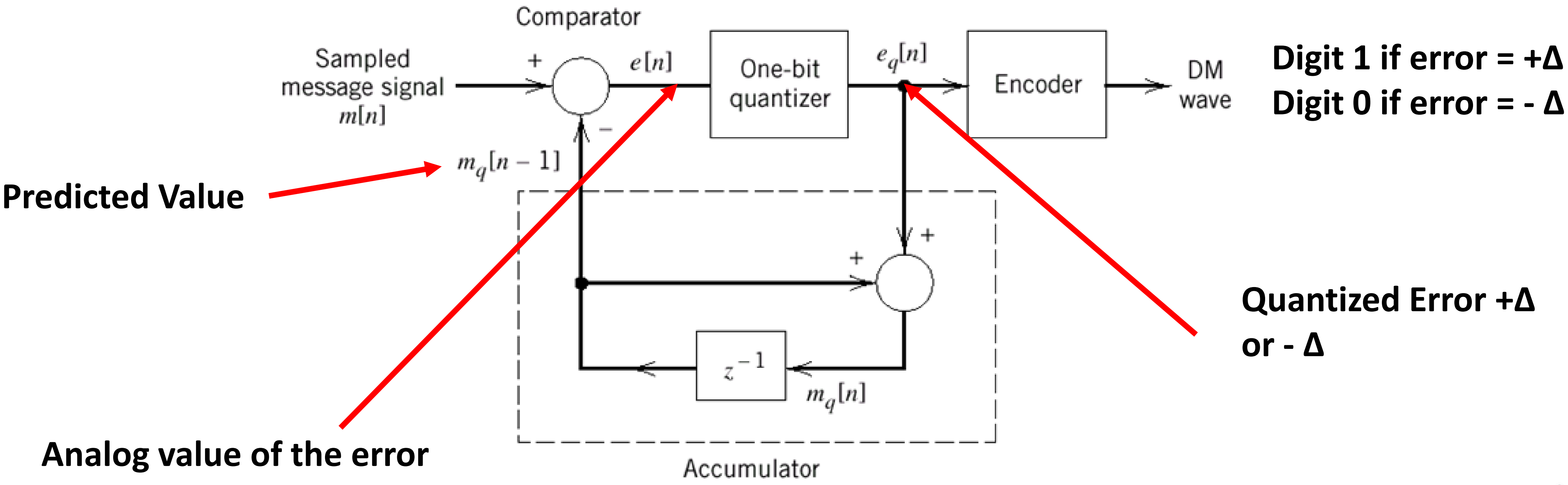
DPCM Receiver

Delta Modulation: Basic Idea



Delta Modulation: The Transmitter Side

- The transmitter side consists of the comparator, the one bit quantizer, the encoder, and the accumulator.
- The accumulator (an integrator) adds the new quantized difference (**+Δ or -Δ**) to the old predicted value to generate the new predicted value.
- The output of the predictor is a staircase approximation of the message signal.



Delta Modulation: The Transmitter Side

Let $m[n] = m(nT_s)$, $n = 0, \pm1, \pm2, \ldots$

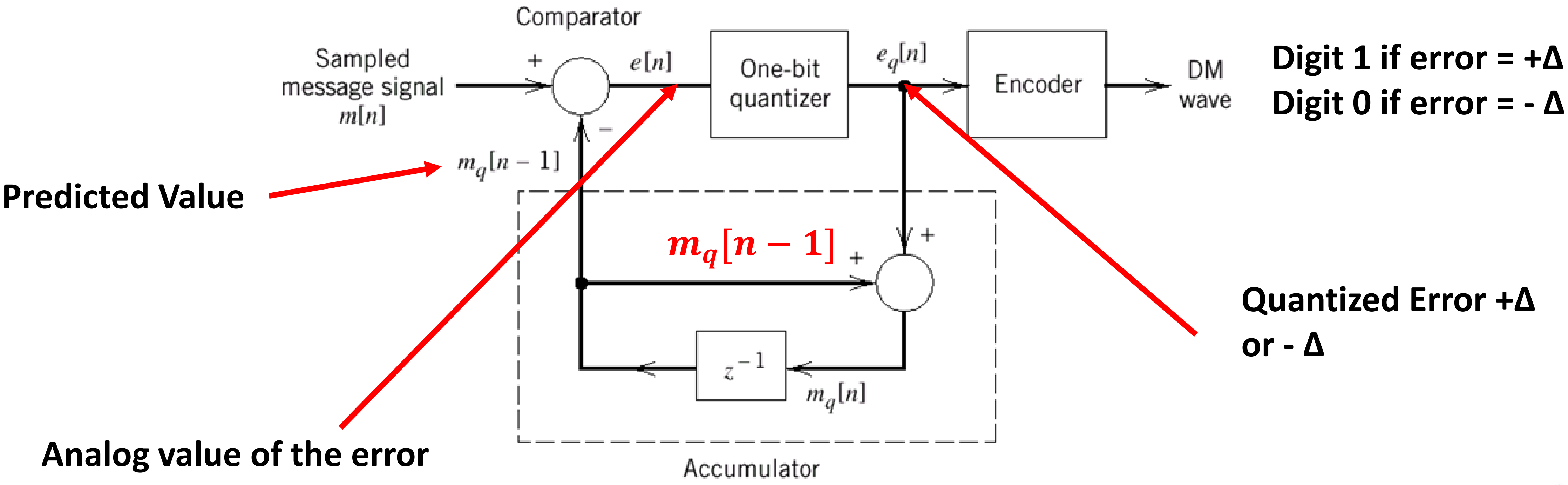
where T_s is the sampling period and $m(nT_s)$ is a sample of $m(t)$. The error signal is

$e[n] = m[n] - m_q[n-1]$

$e_q[n] = \Delta \operatorname{sgn}(e[n])$; quantized error

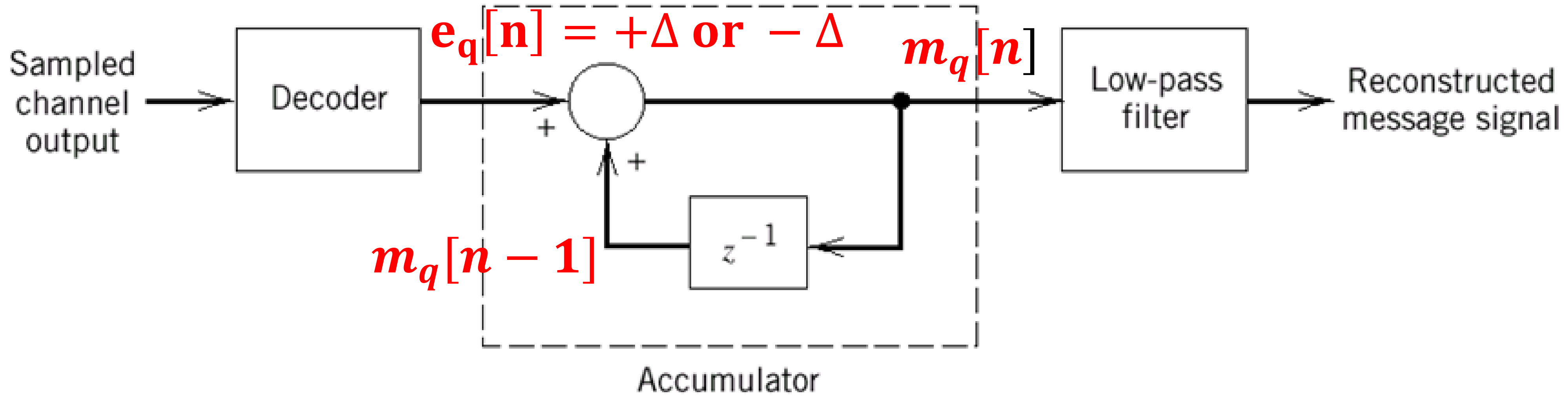
$m_q[n] = m_q[n-1] + e_q[n]$

where $m_q[n]$ is the quantizer output ,$e_q[n]$ is the quantized version of $e[n]$, and Δ is the step size



Delta Modulation: The Receiver Part

- The receiver part consists of the decoder, the accumulator, and a low pass filter.
- The decoder interprets a zero as $-\Delta$ and one as $+\Delta$. These deltas represent the differences between current and previous samples.
- The accumulator regenerates the predicted staircase signal.
- The low pass filter smoothens the predicted signal by removing high frequency components.
- The reconstructed signal $m_q(t)$ is the same as the predicted signal used at the transmitter side

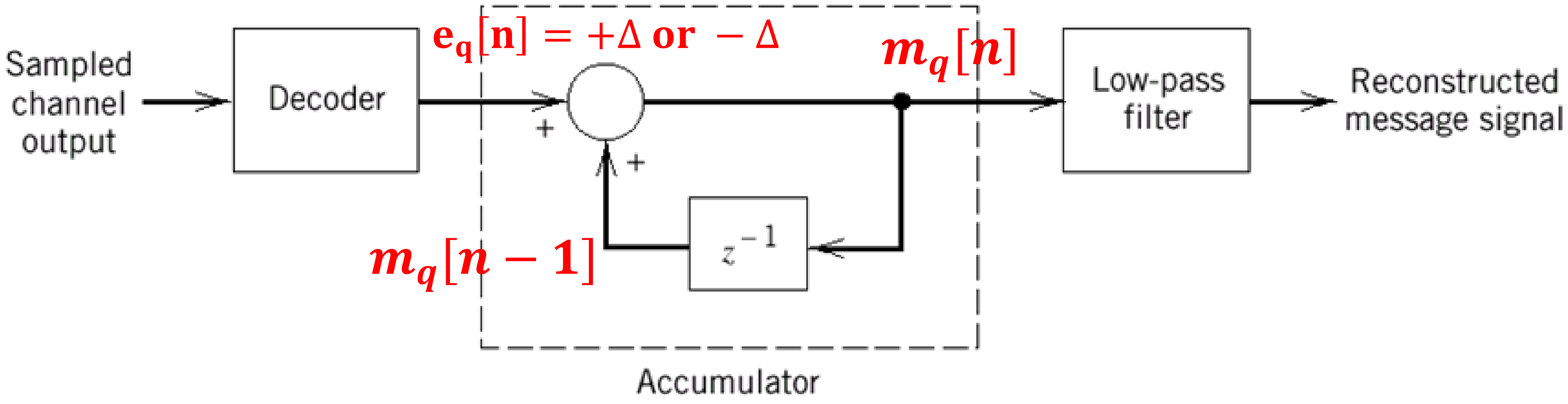


Delta Modulation: The Receiver Part

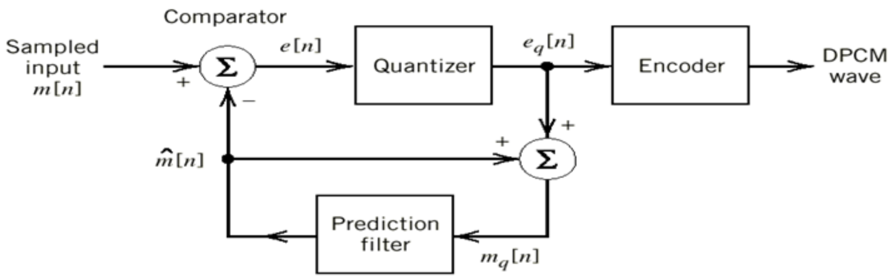
$$m_q[n] = m_q[n-1] + e_q[n]$$

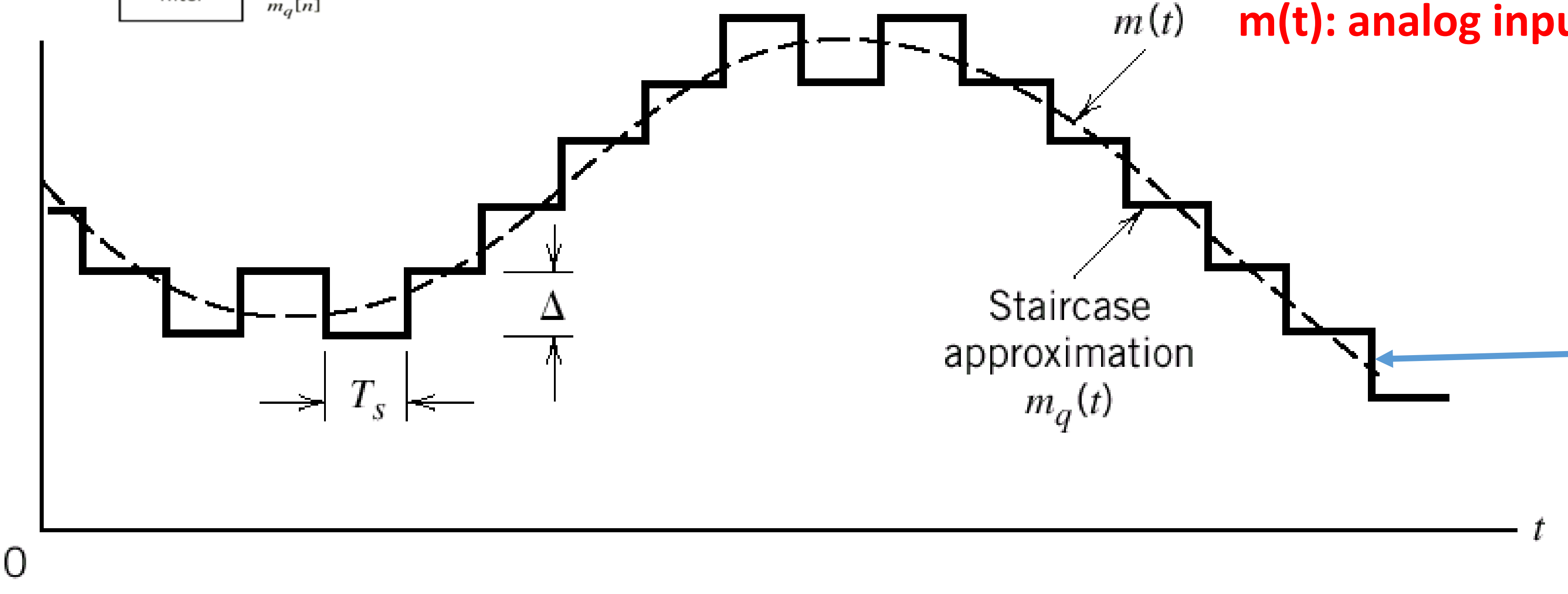
$$\Rightarrow m_q[n] = \Delta \sum_{i=1}^{n} \text{sgn}(e[i])$$

$$= \sum_{i=1}^{n} e_q[i]$$



Delta Modulation: Basic Operation





(a)

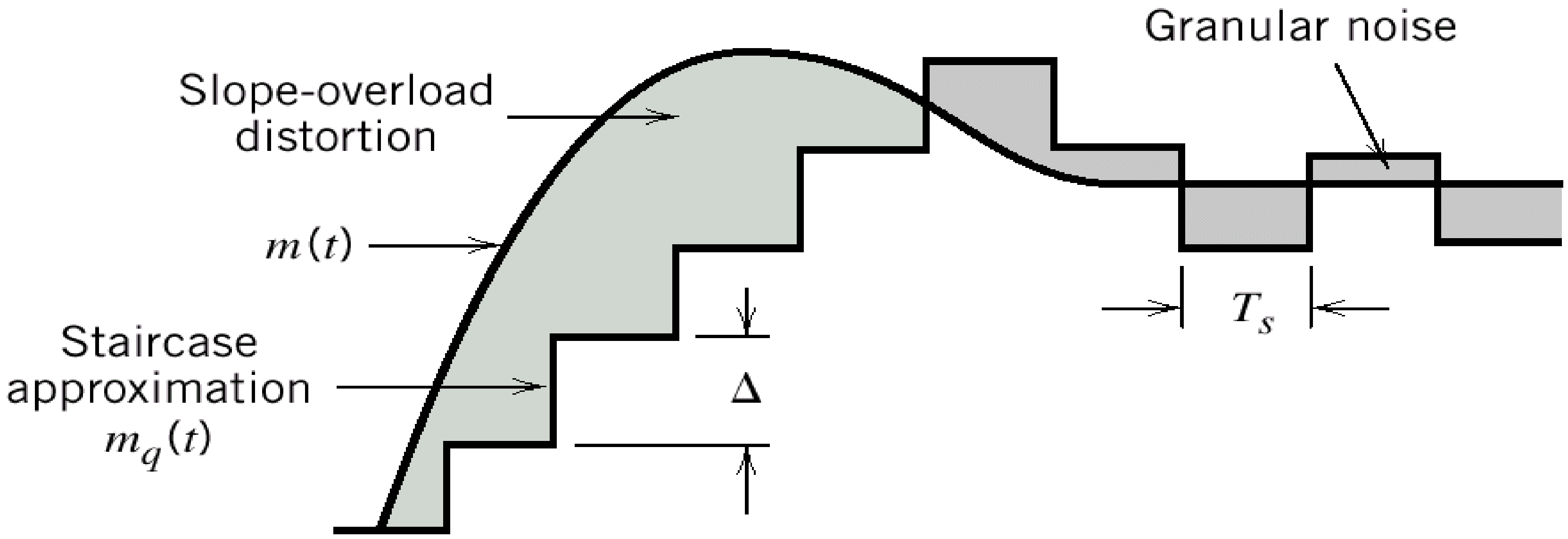
m(t): analog input signal

Δ

Binary sequence at modulator output: 0 0 1 0 1 1 1 1 1 0 1 0 0 0 0 0 0

Slope Overload Distortion and Granular Noise

- **Slope overload** distortion is due to the fact that the staircase approximation $m_q(t)$ can't follow closely the actual curve of the message signal $m(t)$. In contrast to slope-overload distortion, **granular noise** occurs when Δ is too large relative to the local slope characteristics of $m(t)$. granular noise is similar to quantization noise in PCM.
- It seems that a large Δ is needed for rapid variations of $m(t)$ to reduce the slope-overload distortion and a small Δ is needed for slowly varying $m(t)$ to reduce the granular noise. The optimum Δ can only be a compromise between the two cases.
- To satisfy both cases, an adaptive DM is needed, where the step size Δ can be adjusted in accordance with the input signal $m(t)$ (not to be covered in this lecture)



Slope Overload

Slope overload occurs when the signal changes at a rate faster than that of the predicted signal. To avoid slope overload, we must have

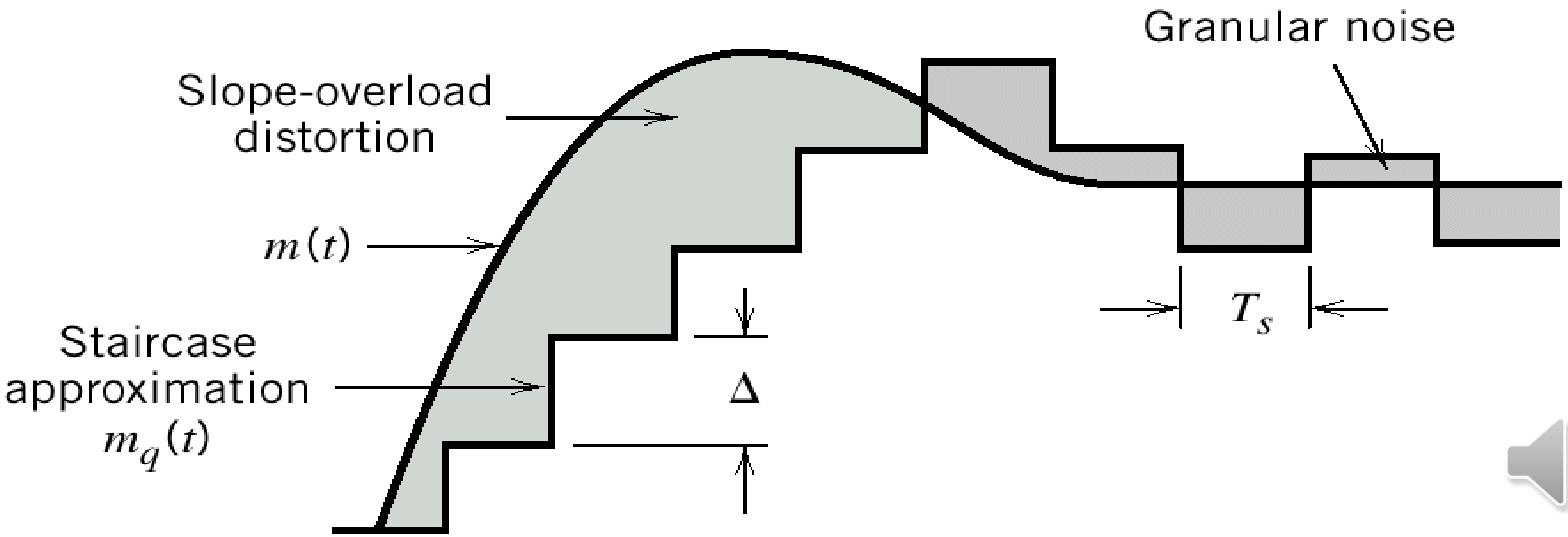
$$\frac{\Delta}{T_s} \geq max\left|\frac{dm(t)}{dt}\right|$$

When $\mathrm{m(t)} = A_m \cos(2\pi f_m t)$, the condition for avoiding slope overload becomes

$$\frac{\Delta}{T_s} \geq 2\pi A_m f_m \; ; \quad \text{OR} \quad \Delta \geq 2\pi T_s A_m f_m$$

As we can see, slope overload depends on three factors:

- Sampling frequency (larger sampling, reduces the effect)
- Message amplitude (larger amplitude, increases the effect)
- Message frequency (larger message frequency, increases the effect

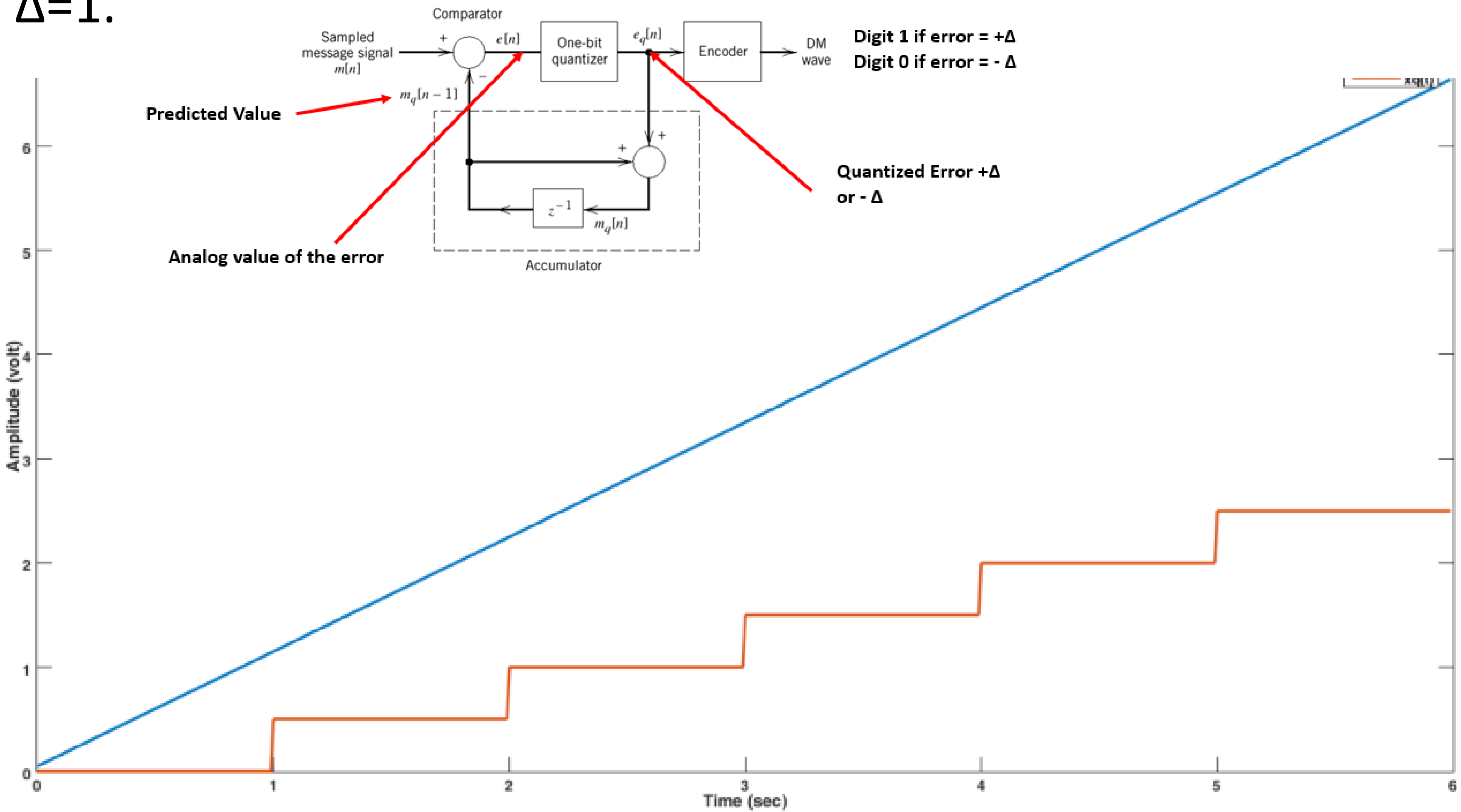


Adaptive Delta Modulation

- The step size in delta modulation affects the quality of the transmitted waveform (slope overload or granular noise).
- A larger step-size is needed in the steep slope of modulating signal
- a smaller step size is needed where the message has a small slope
- In adaptive delta modulation, the step size is adjusted via a feedback control signal so as to reduce both slope overload and granular noise effects.
- ADM quantizes the difference between the value of the current sample and the predicted value of the next sample. It uses a variable step height to predict the next values, for the faithful reproduction of the fast varying values.

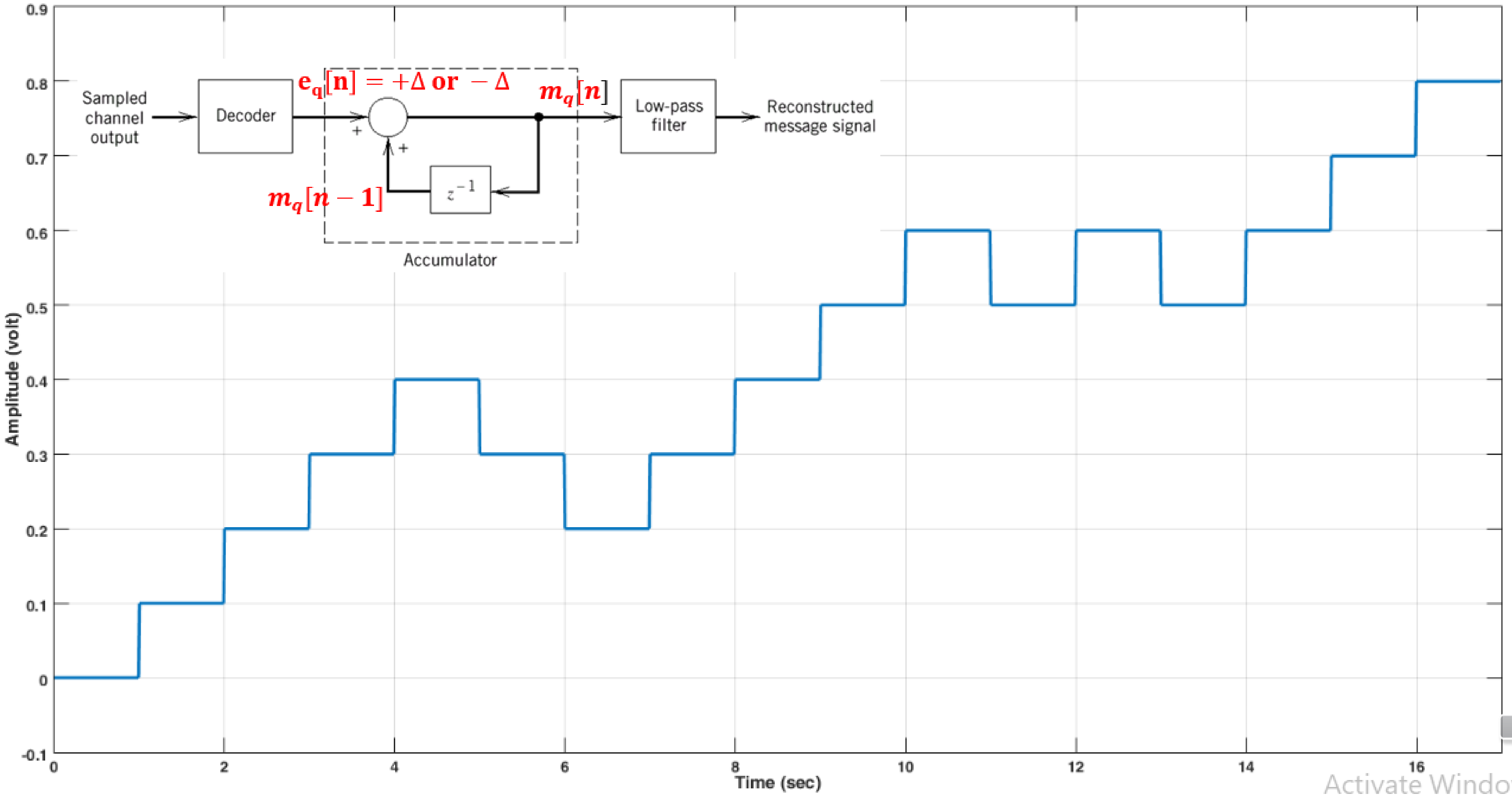
Delta Modulation: Example

- Draw the output of the DM given that the input corresponds to $x(t) = 1.1t + 0.05$ when the input is sampled at t = 0, 1, 2, 3, 4, 5, … and Δ=1.



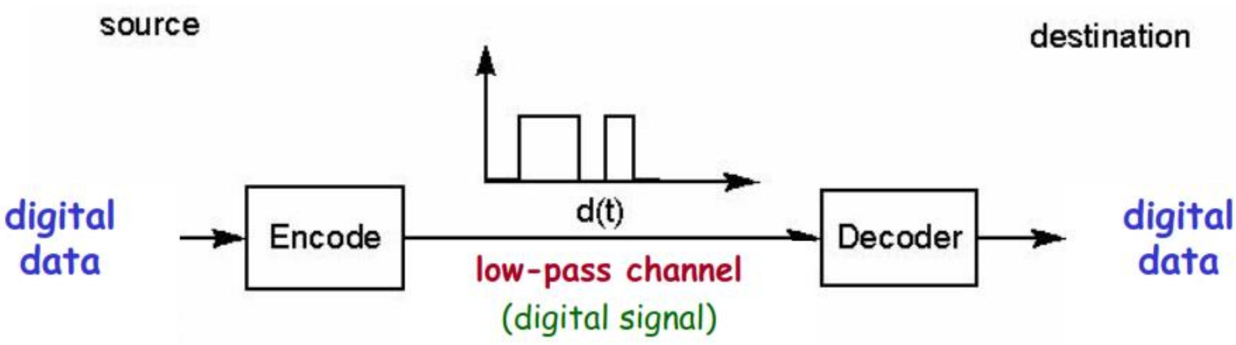
Delta Demodulation: Example

- Reconstruct a staircase signal at the receiver side of a delta demodulator with Δ = 0.1V, when the received data sequence is 1 1 1 1 0 0 1 1 1 1 0 1 0 1 1 1.



Line Encoding

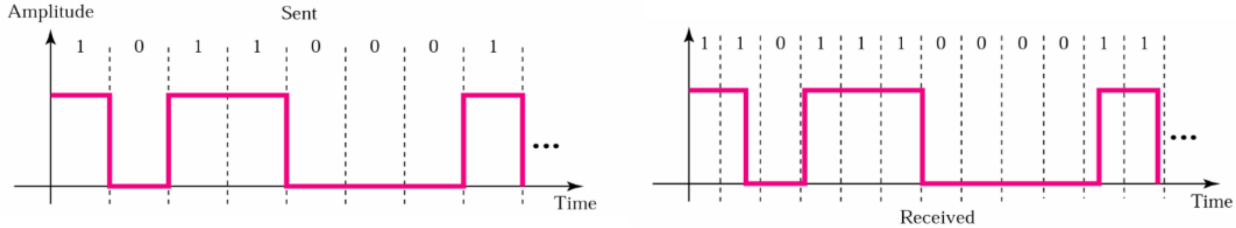
- The assignment of pulses (an electrical signal) to the binary digits that come out of the PCM or DPCM system.
- Line coding encodes the bit stream for transmission through a line, or a cable.
- It is used for communications between the CPU and peripherals, and for short-distance baseband communications, such as the Ethernet.



Two Design Considerations

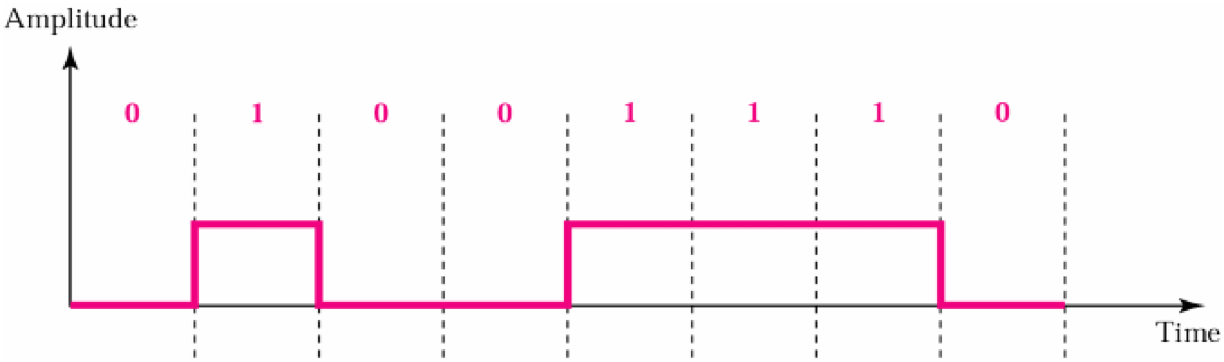
- **DC Component in Line Coding:** Some line coding schemes have a residual (DC) component, which is generally undesirable.
  - Transformers do not allow passage of DC component.
  - DC component $\Rightarrow$ extra energy – useless!

- **Self-Synchronization (clocking):** To correctly interpret signal received from sender, receiver's bit interval must exactly correspond to sender's bit intervals
  - If receiver clock is faster/slower, bit intervals not matched $\Rightarrow$ receiver misinterprets signal
  - Self-synchronizing digital signals include timing information in itself, to indicate the beginning & end of each pulse



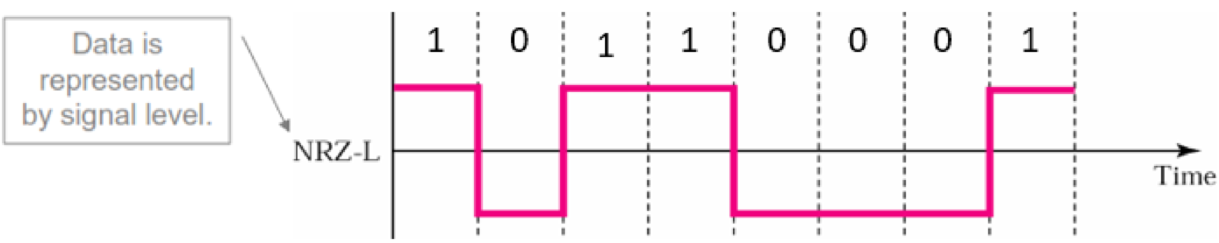
Unipolar Line Coding (Unipolar non-return to zero)

- Uses only one non-zero and one zero voltage level to represent binary digits 1 and 0
- Simple to implement, but obsolete due to two main problems:
  - Presence of a DC component.
  - Lack of synchronization for long series of 1-s or 0-s



Polar Line Coding: Polar non-return to zero

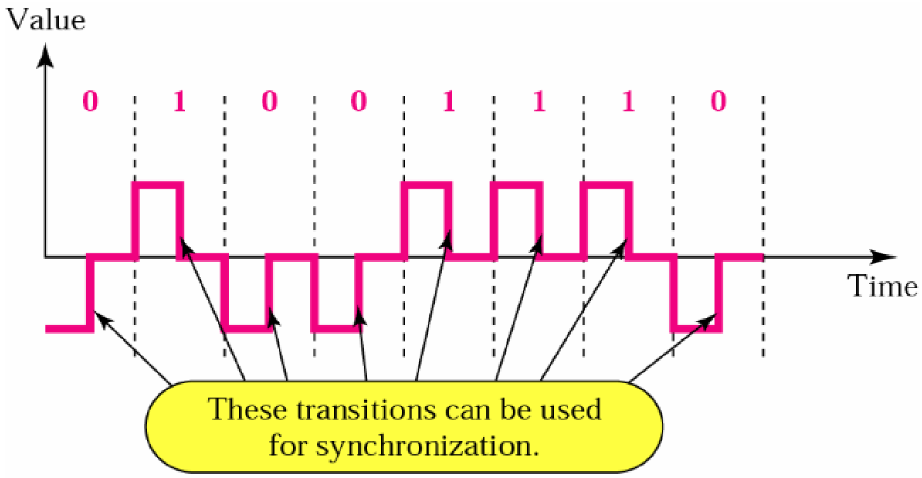
- Uses two non-zero voltage level to represent digits 1 and digit 0. +ve for 1 and –ve for 0
- No DC component is present
- Poor synchronization for long series of 1-s or 0-s



Polar Non-return to Zero

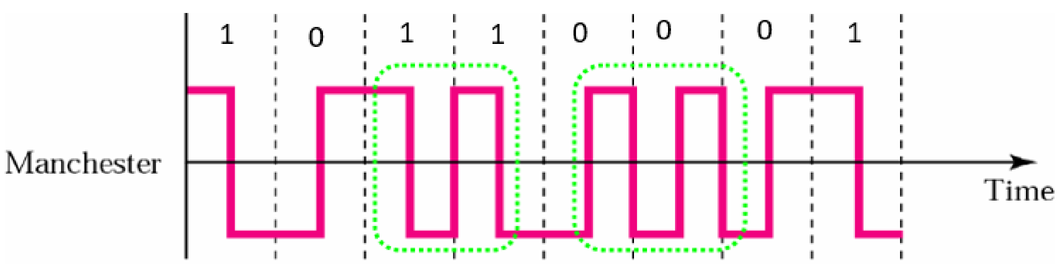
Polar Line Coding: Polar return to zero

- Uses two non-zero voltage level to represent digits 1 and digit 0. +ve for 1 and –ve for 0. **Must** return to zero halfway through each bit interval.
- No DC component is present.
- Perfect synchronization for long series of 1-s or 0-s.
- Twice the bandwidth required for polar non-return to zero, $B.W \propto \frac{1}{pulse\ width}$.



Manchester Line Coding

- Inversion at the middle of each bit interval is used for both synchronization and bit representation
- Digit 0 ⇒ neg-to-pos transition, 1 ⇒ pos-to-neg transition
- Perfect synchronization for long series of 1-s or 0-s
- There is always transition at the middle of the bit, and maybe one transition at the end of each bit.
- Fine for alternating sequences of bits (10101), but wastes bandwidth for long runs of 1-s or 0-s.
- Used by IEEE 802.3 (Ethernet).
- No DC component is present.
- Twice the bandwidth required for polar non-return to zero. Two pulses are used to represent one bit.

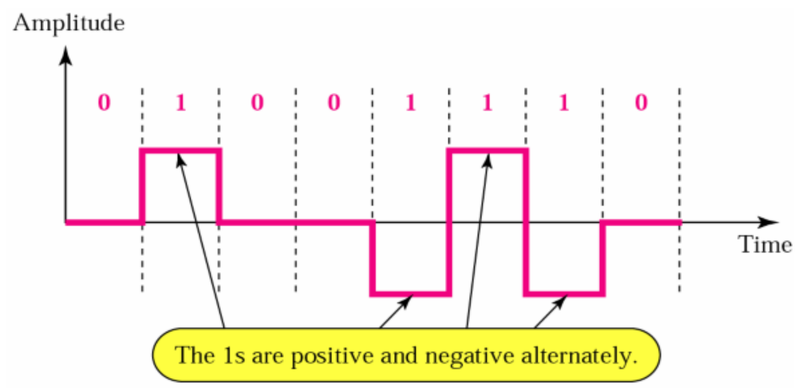


Manchester Encoding

Bipolar Line Coding

- Uses two non-zero and zero voltage level for representation of two data levels.
- **0 = zero level; 1 = alternating positive and negative level.**

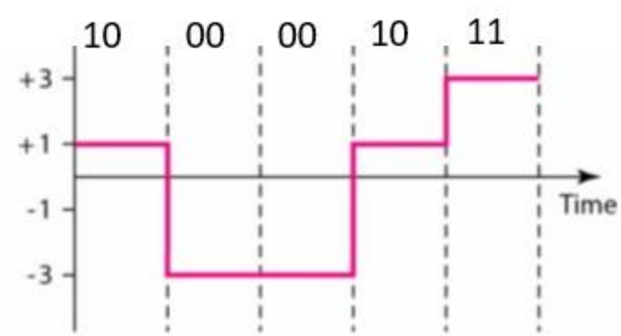
- If first bit 1 is represented by a positive amplitude, second will be represented by negative amplitude, third by positive, etc.
- Less bandwidth required than with Manchester coding (for any sequence of bits).
- Loss of synchronization is possible for long runs of 0-s.
- No DC component is present.



2B1Q (2 Bipolar to 1Quaternary) Line Coding

- Data patterns of size 2 bits are encoded as one signal element belonging to a four-level signal.
- Data is sent two time faster than with polar non-return to zero.
- Receiver has to discern 4 different thresholds

Binary Input	Output Voltage
00	-3
01	-1
10	1
11	3



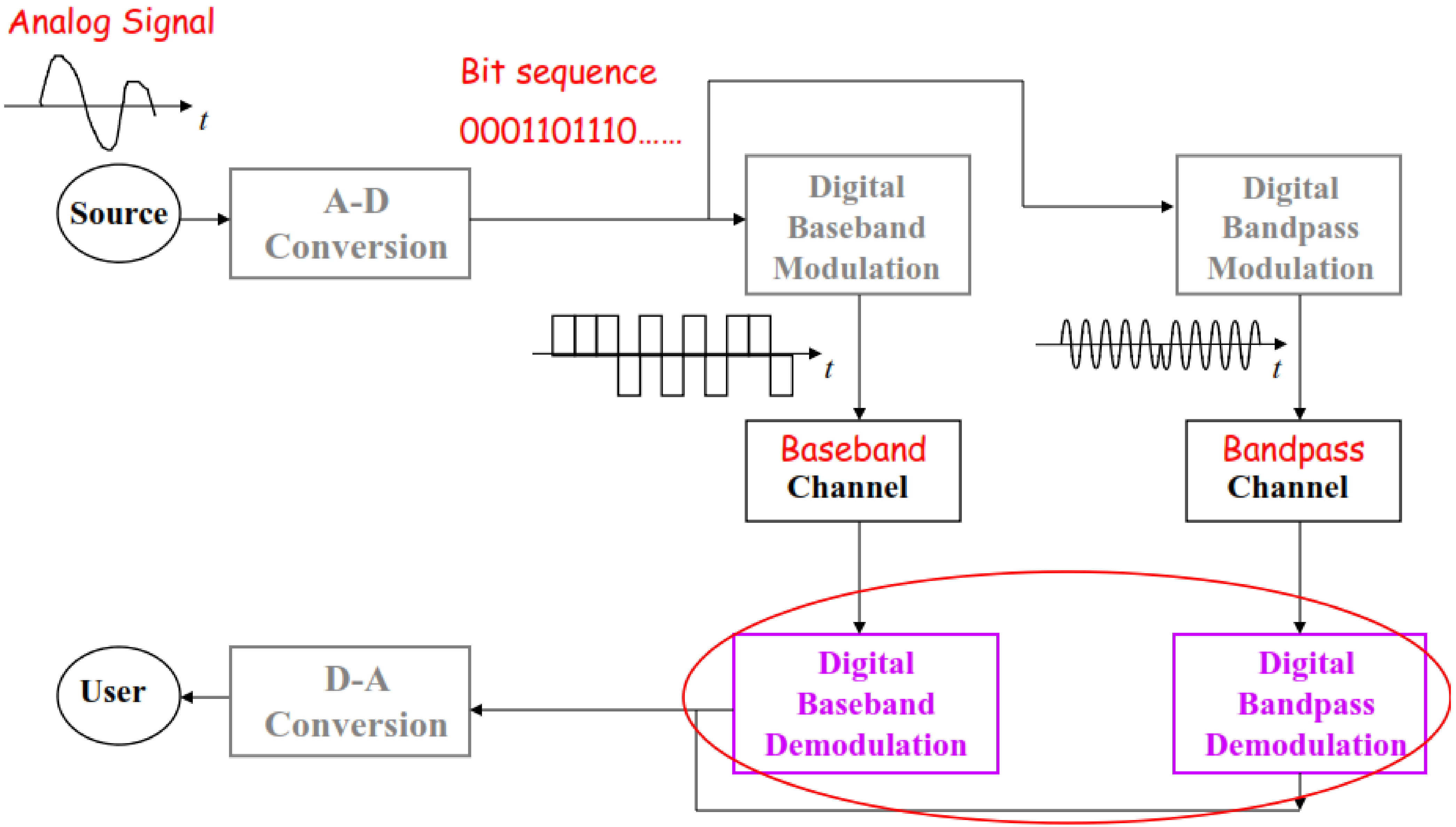
Optimum Receiver and Digital Binary Transmission

In binary data transmission over a communication channel, logic 1 is represented by a signal $s_1(t)$ and logic 0 by a signal $s_2(t)$. The time allocated for each signal is the symbol duration τ, where τ is related to the data rate by $r_b = 1/\tau$. We have two types of data transmission:

Baseband Data Transmission: Binary data transmission by means of two baseband waveforms (typically, two voltage levels) is referred to as baseband signaling. The spectrum of the transmitted signal occupies the low part of the frequency band (around the zero frequency). No high frequency carrier is used in this mode of transmission.

Bandpass Data Transmission: The baseband data modulates a high frequency carrier to produce a modulated signal, whose spectrum is centered around the carrier frequency.

Digital Communication Block Diagram

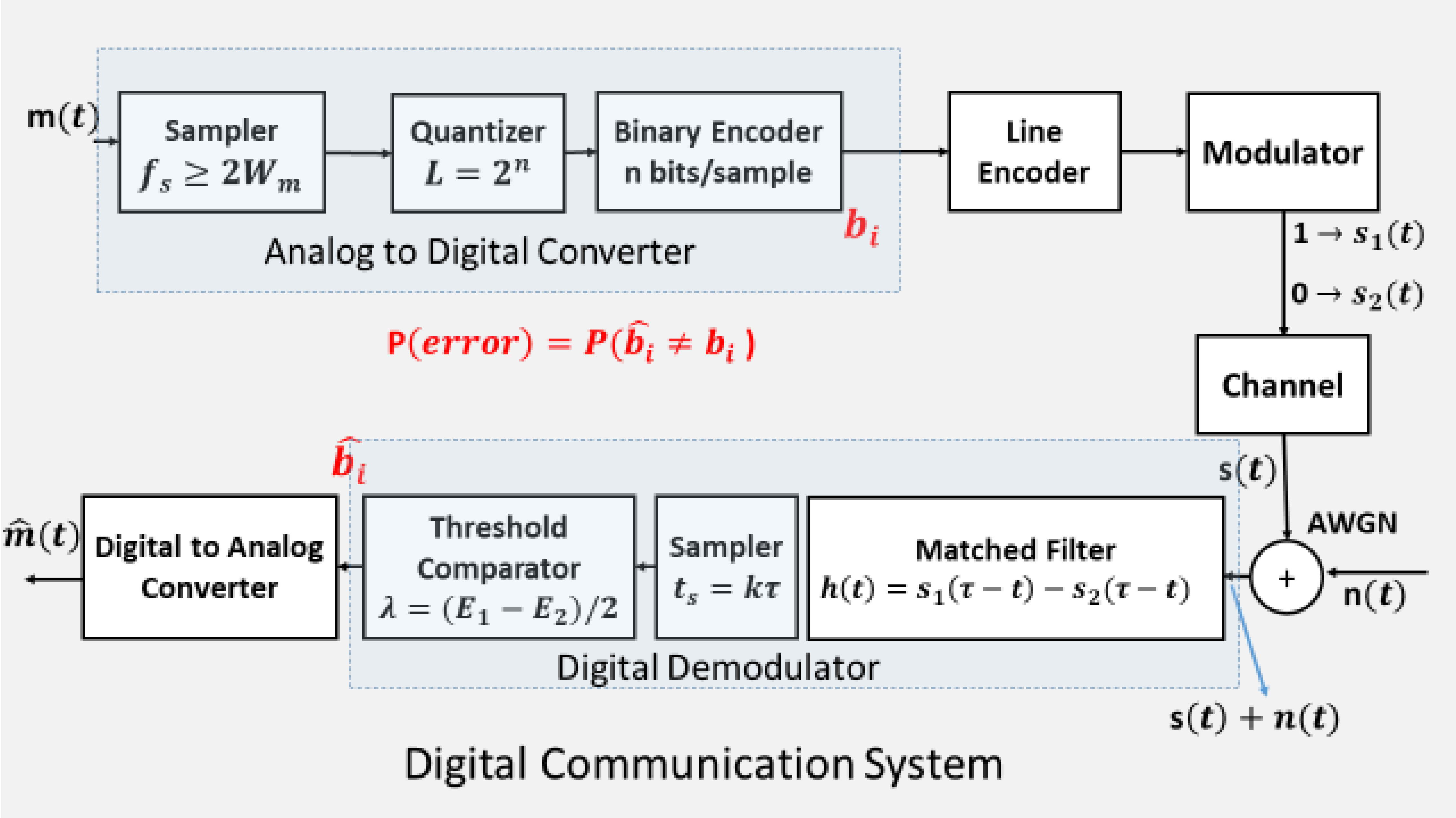


Assumptions

- The channel noise $n(t)$ is additive white Gaussian (AWGN) with a double-sided PSD of $N_0/2$. Noise is assumed to be added at the front end of the receiver.
- The data component at the front end of the receiver is assumed to be an exact replica of the transmitted signal, in the sense that the transmission bandwidth of the medium is wide enough to reproduce the signal without distortion.
- Bits in different time intervals are assumed independent.
- The signal to be processed by the receiver is the noisy signal $\boldsymbol{y(t) = s_i(t) + n(t)}$

Based on $\boldsymbol{y(t)}$, the task of the receiver is to decide whether a 1 or a 0 was transmitted during each transmission slot $\boldsymbol{\tau}$ with minimum probability of error.

Transmitter and Receiver Sides in Digital Communication System

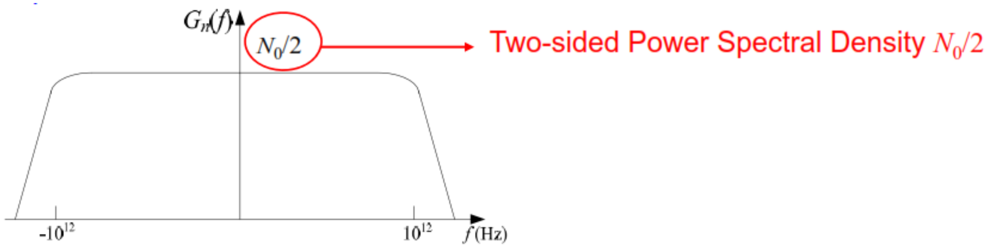


Thermal Noise

- **Thermal Noise**: caused by the random motion of electrons within electronic devices
- Thermal noise, $n(t)$, is modeled as a wise sense stationary (WSS) Gaussian random process.

  wide

- The thermal noise has a power spectrum that is constant from dc to approximately 10^{12} Hz; hence, $n(t)$ can be approximately regarded as a white process.
- Thermal noise is superimposed (added) on the transmitted signal. The received signal is $y(t) = s(t) + n(t)$.

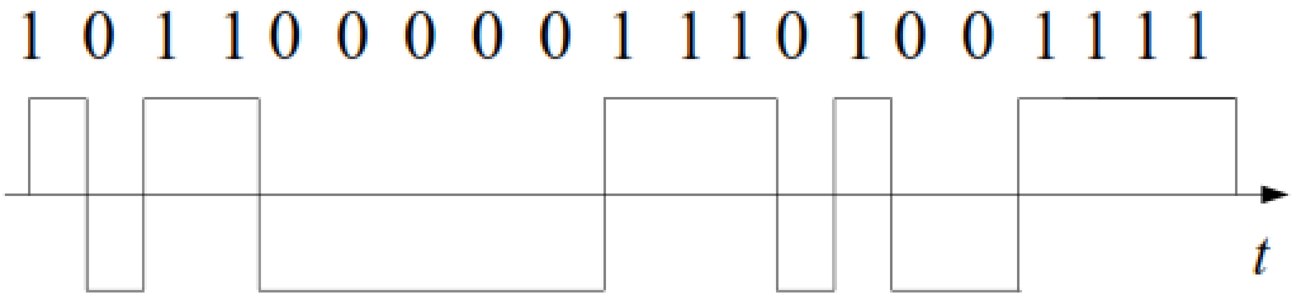


- The mean value of the thermal noise $n(t)$ is zero.
- At any given time t_0 the probability density function of $n(t_0)$ follows the Gaussian distribution; $N(0, \sigma_0^2)$; where $\sigma_0^2 = E(n(t))^2$ is the noise power.

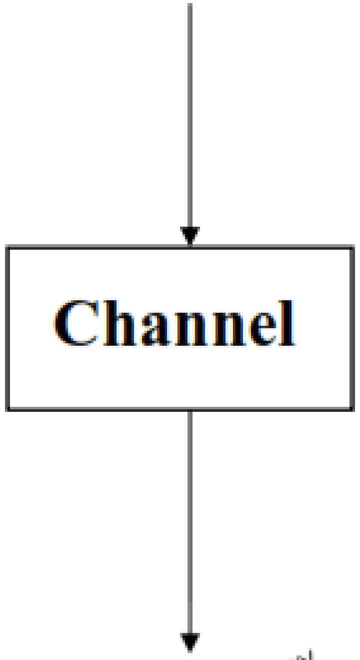
$$f_n(n) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-n^2/2\sigma_0^2}$$

Effect of Noise and Channel on Received Data

Transmitted signal
$s(t)$

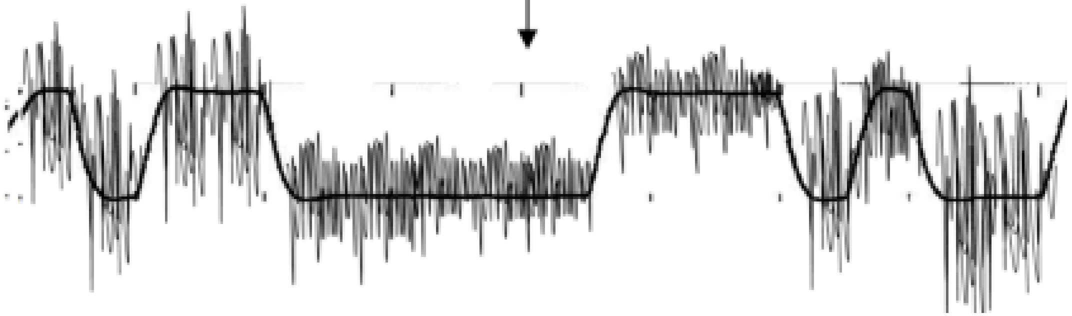


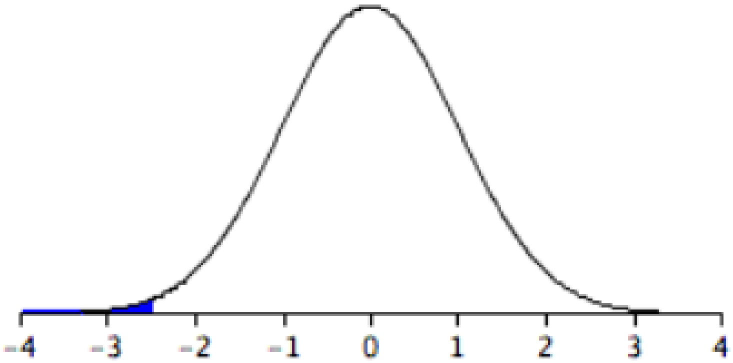
- Suppose that channel bandwidth is properly chosen such that most frequency components of the transmitted signal can pass through the channel.



- Thermal Noise: caused by the random motion of electrons within electronic devices.

Received signal
$y(t)$





Basic Elements of the Receiver

To decide on whether logic 1 or logic 0 was transmitted during a given time slot τ, the received signal (transmitted signal and noise) passes through three basic units.

Filter: The optimum filter, which we will also call the matched filter.

Sampler: Samples the received signal (data component plus noise) at some time $t = t_0 = \tau = symbol\ duration.$

Threshold comparator: If the sampled value is larger than a given threshold, λ, digit 1 is declared true, otherwise digit 0 is declared true.

There are three design elements at the receiver

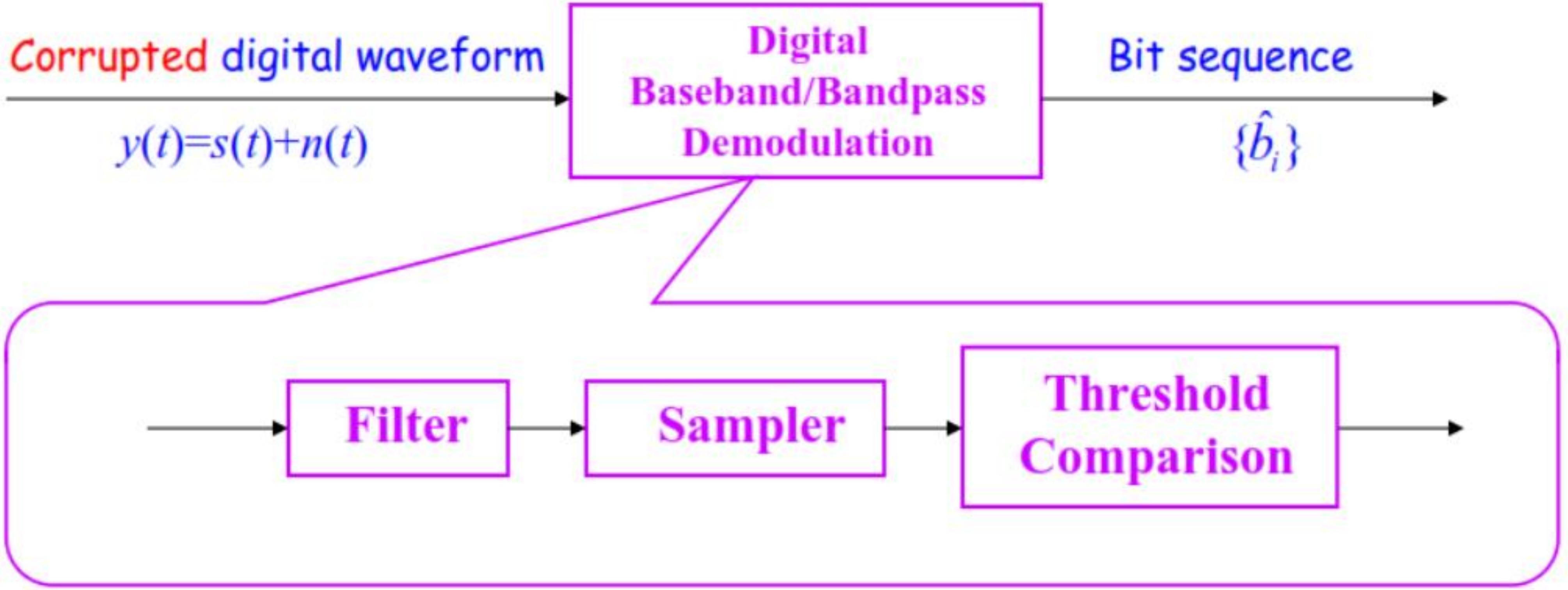
a. The impulse response $h(t)$ of the filter
b. The sampling time t_0
c. The threshold λ

Basic Elements of the Receiver

These parameters should be chosen so as to minimize the average probability of error (or bit error rate BER), defined as

$$P_b = P(b_i \neq \hat{b}_i)$$

$$P_b = \Pr\{\hat{b}_i = 1, b_i = 0\} + \Pr\{\hat{b}_i = 0, b_i = 1\}$$



Optimum Receiver and Digital Binary Transmission

Transmitted signal $s(t)$

Received signal $y(t)=s(t)+n(t)$

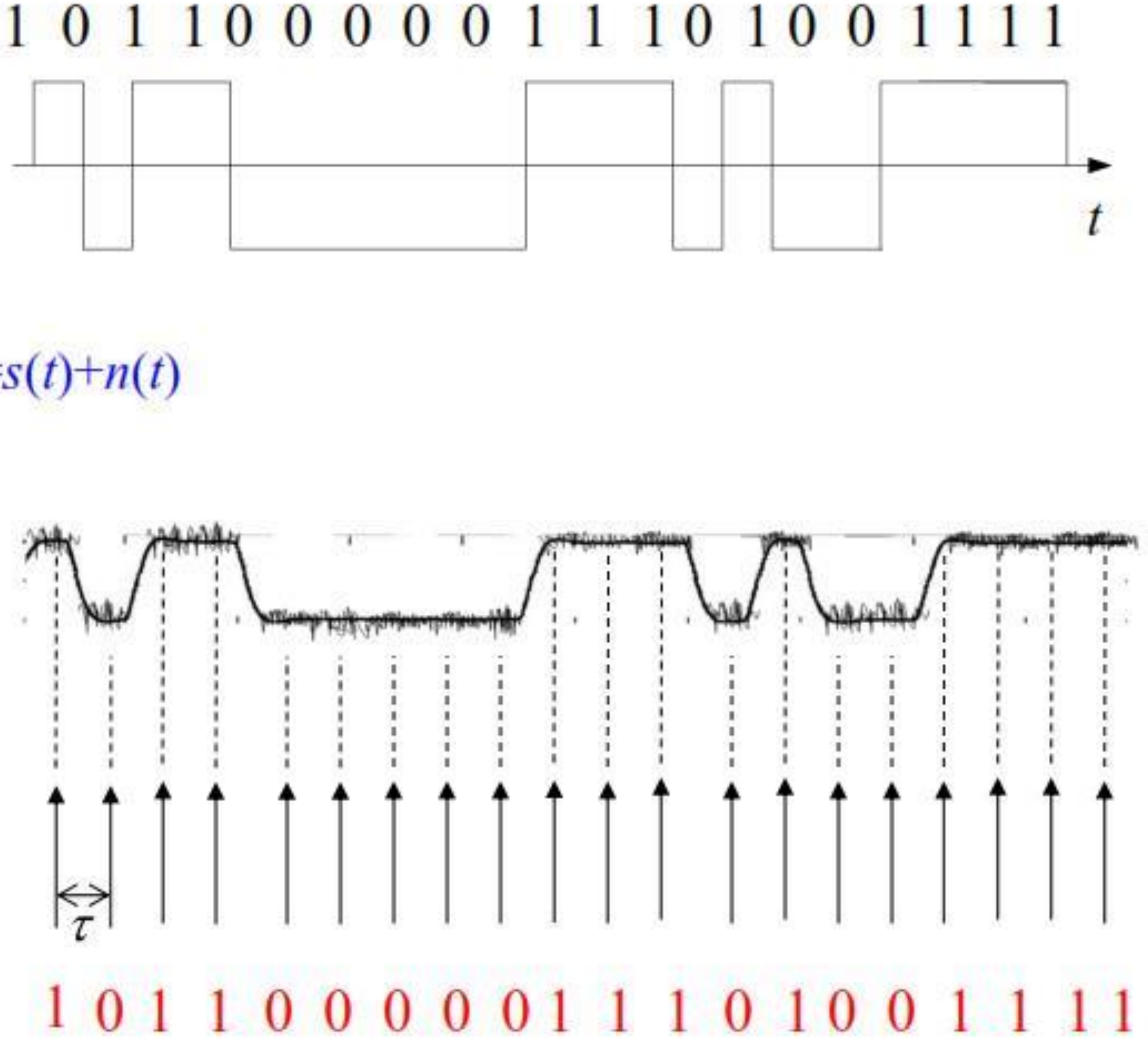
Step 1: Filtering

Step 2: Sampling

Step 3: Threshold Comparison

Sample>0 ⇨ 1

Sample<0 ⇨ 0



Theorem on the Optimum Binary Receiver

Consider a binary communication system, corrupted by AWGN with power spectral density $N_0/2$, where the equally probable binary digits 1 and 0 are represented by the signals $s_1(t)$ and $s_2(t)$, respectively. The transmission time for each signal is τ sec. The optimum receiver elements, i.e., the elements that minimize the receiver probability of error are given by

Impulse response of the matched filter: $\boldsymbol{h(t) = s_1(\tau - t) - s_2(\tau - t)}$, $0 \le t \le \tau$

Optimum sampling time: $\boldsymbol{t_s = \tau}$

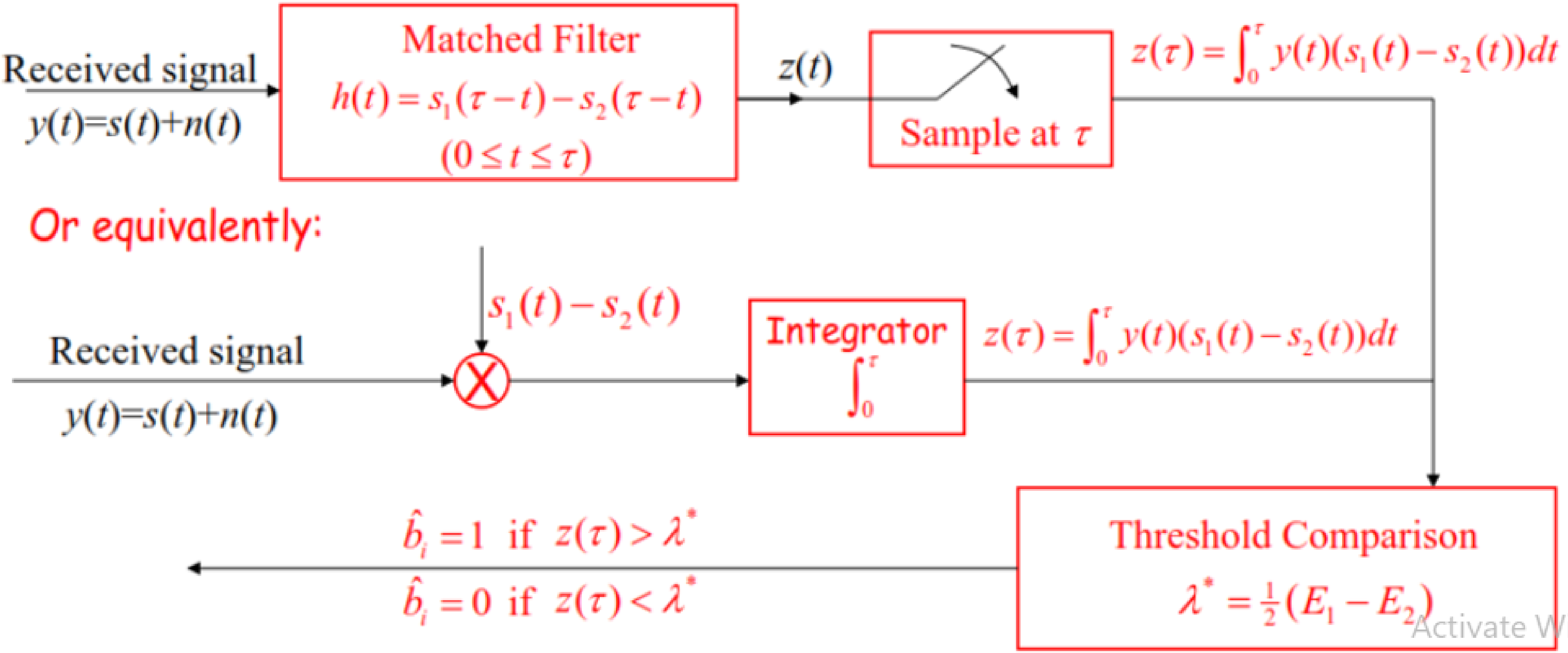
Optimum threshold of comparator: $\lambda^* = \frac{1}{2}(\boldsymbol{E_1 - E_2})$, $E_k = \int_0^\tau (s_k(t))^2 dt$, $k = 1,2$

When these elements are used, the system minimum probability of error is

$$\boldsymbol{P_b^* = Q\left(\sqrt{\frac{\int_0^\tau (s_1(t) - s_2(t))^2 dt}{2N_0}}\right)}$$

Theorem on the Optimum Binary Receiver

The structure of the optimum receiver is depicted in the figure below. Note that the receiver can be implemented in terms of the matched filter and, equivalently, in terms of a correlator (a multiplier followed by an integrator).

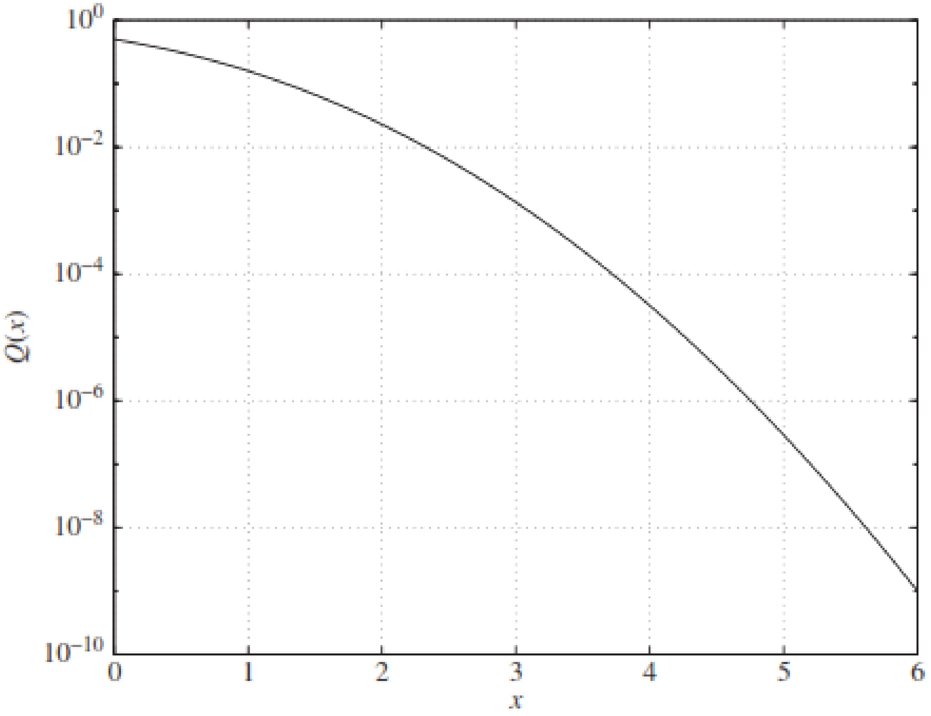


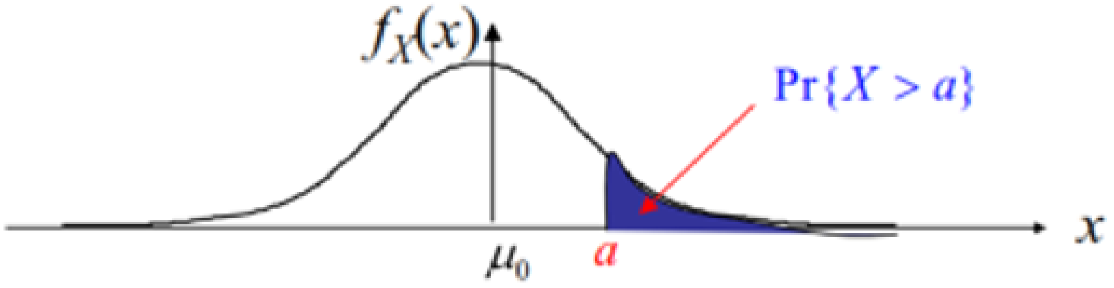
The Q-Function

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

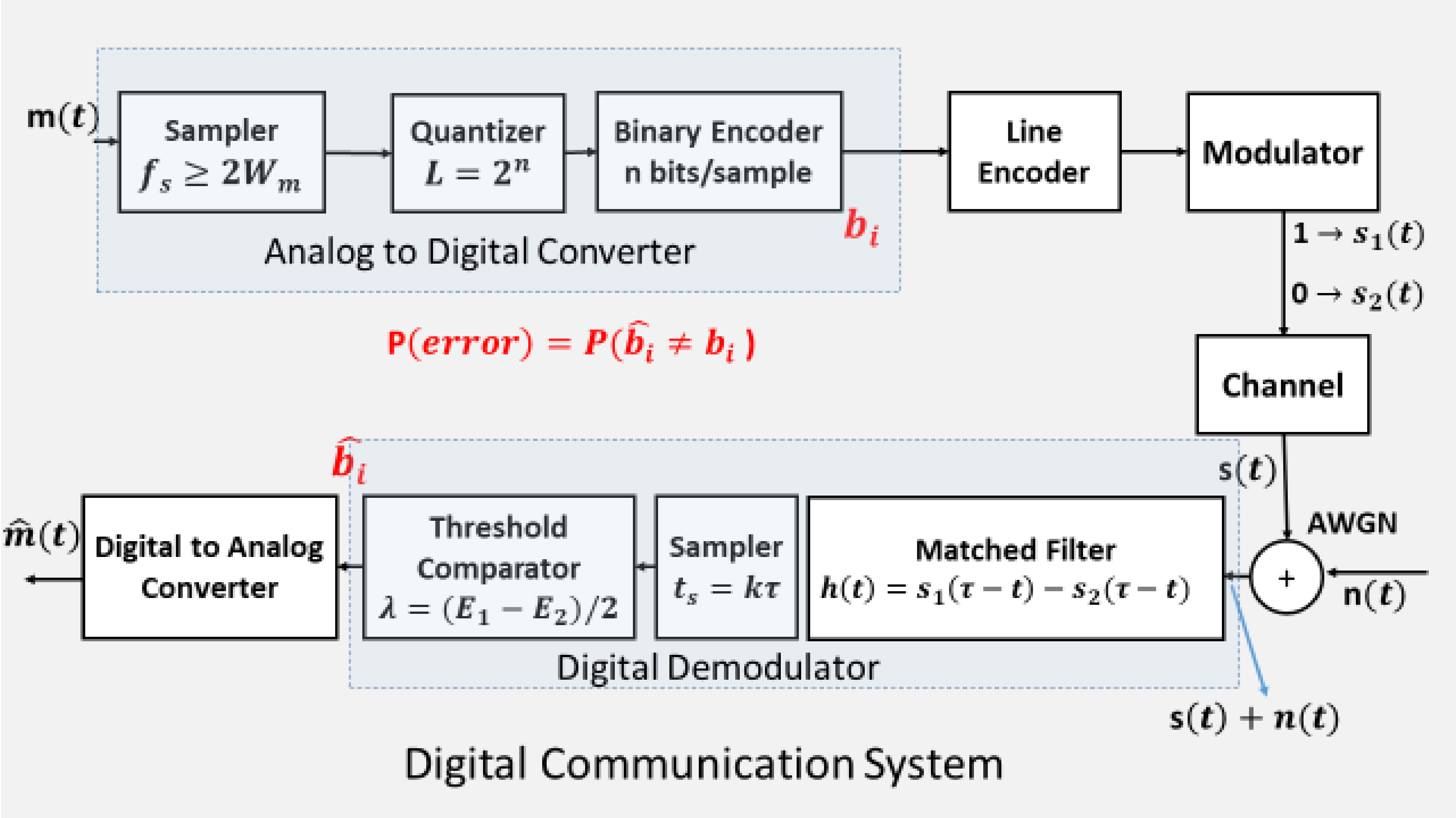
- $Q(\alpha)$ is a decreasing function of α.
- For $X \sim \mathcal{N}(\mu_0, \sigma_0^2)$,

$$\Pr\{X > a\} = \int_a^{\infty} f_X(x)dx = \int_a^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right\} dx = Q\left(\frac{a-\mu_0}{\sigma_0}\right)$$

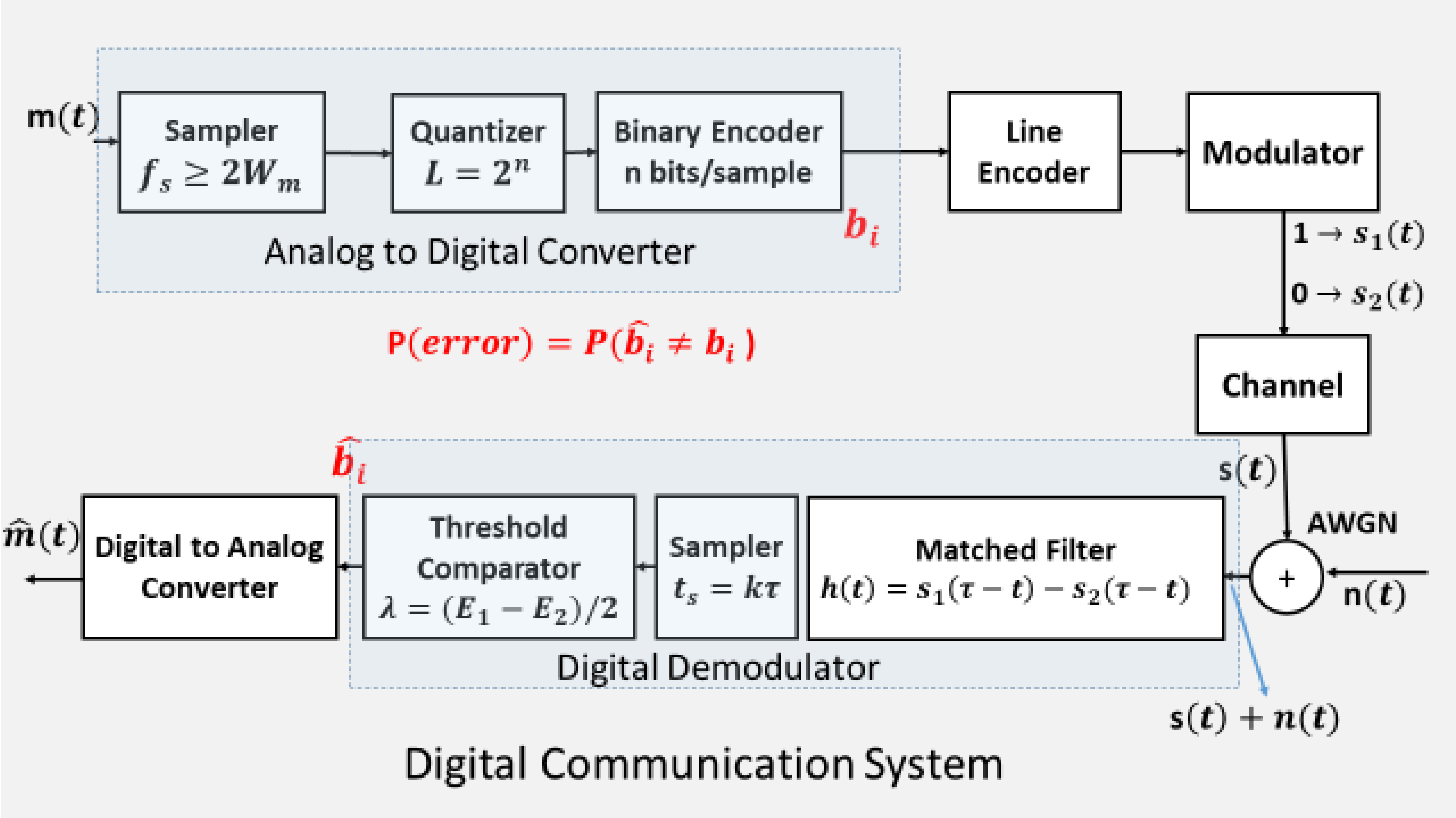




Transmitter and Receiver Sides in Digital Communication System



Matched Filter and Performance of the Optimum Receiver



Theorem on the Optimum Binary Receiver

Consider a binary communication system, corrupted by AWGN with power spectral density $N_0/2$, where the equally probable binary digits 1 and 0 are represented by the signals $s_1(t)$ and $s_2(t)$, respectively. The transmission time for each signal is τ sec. The optimum receiver elements, i.e., the elements that minimize the receiver probability of error are given by

Impulse response of the matched filter: $\boldsymbol{h(t) = s_1(\tau - t) - s_2(\tau - t)}$, $0 \le t \le \tau$

Optimum sampling time: $\boldsymbol{t_s = \tau}$

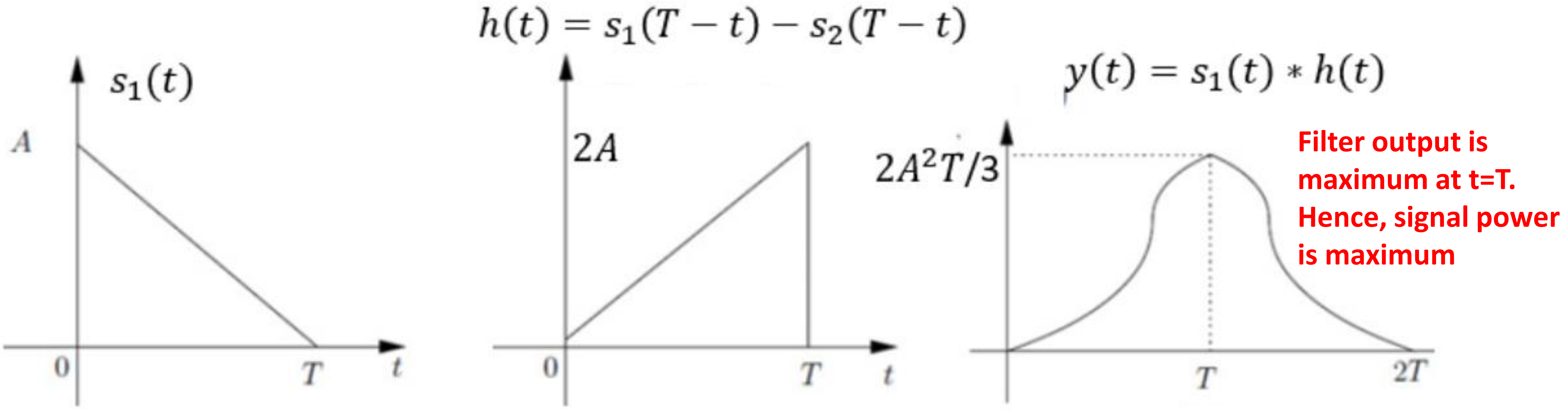
Optimum threshold of comparator: $\lambda^* = \frac{1}{2}(\boldsymbol{E_1 - E_2})$, $E_k = \int_0^\tau (s_k(t))^2 dt\,, k = 1,2$

When these elements are used, the system minimum probability of error is

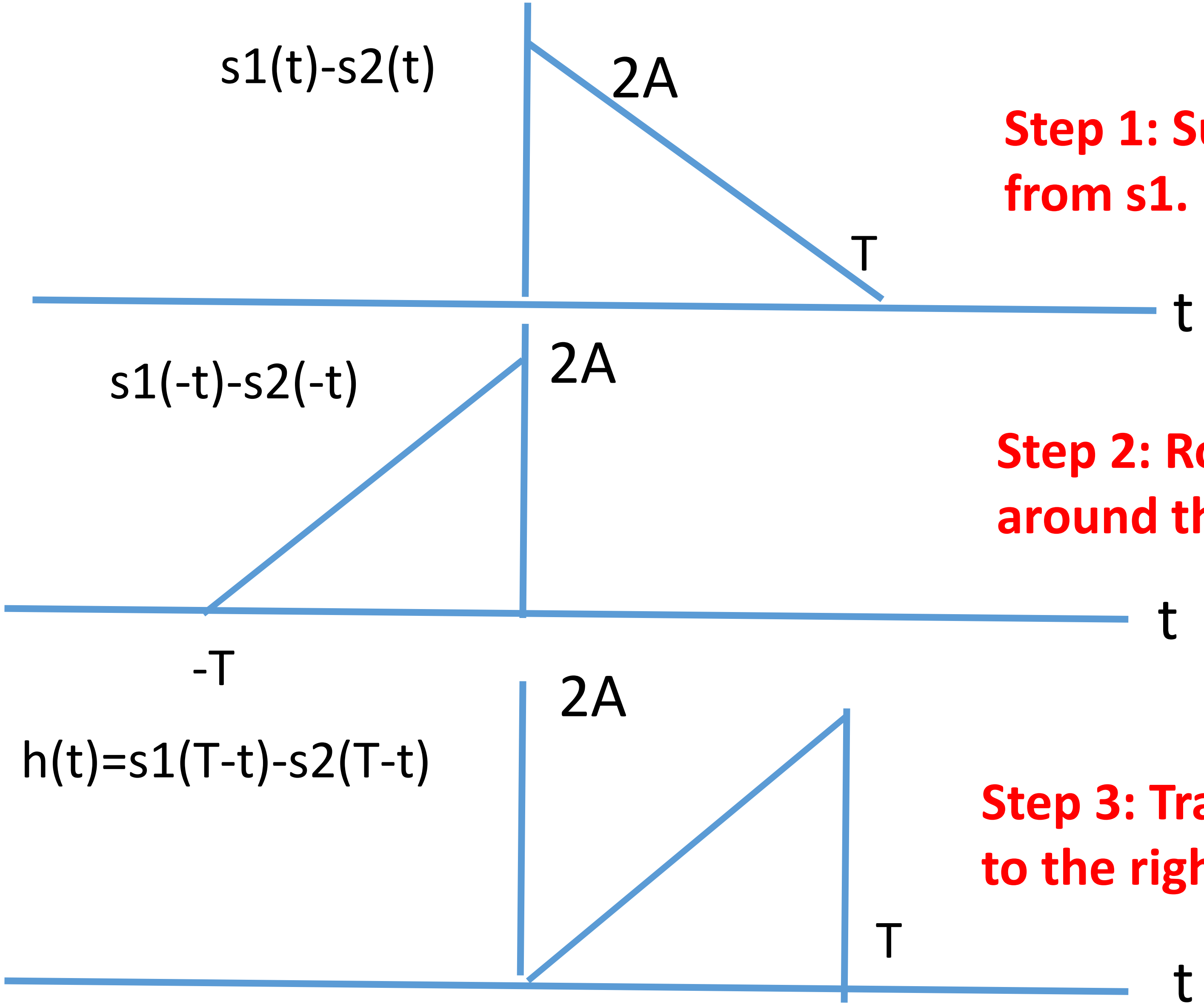
$$\boldsymbol{P_b^* = Q\left(\sqrt{\frac{\int_0^\tau (s_1(t) - s_2(t))^2 dt}{2N_0}}\right)}$$

Output of a Matched Filter

Example 2: The next figure shows a signaling scheme where $s_2(t) = -s_1(t)$. The impulse response of the matched filter is $h(t) = s_1(T-t) - s_2(T-t)$. The figure shows the filter output when $s_1(t)$ is applied to the filter. Note that the output attains its maximum value at time t=T, which is the sampling time chosen to maximize the output SNR.



Matched Filter Derived from Signals



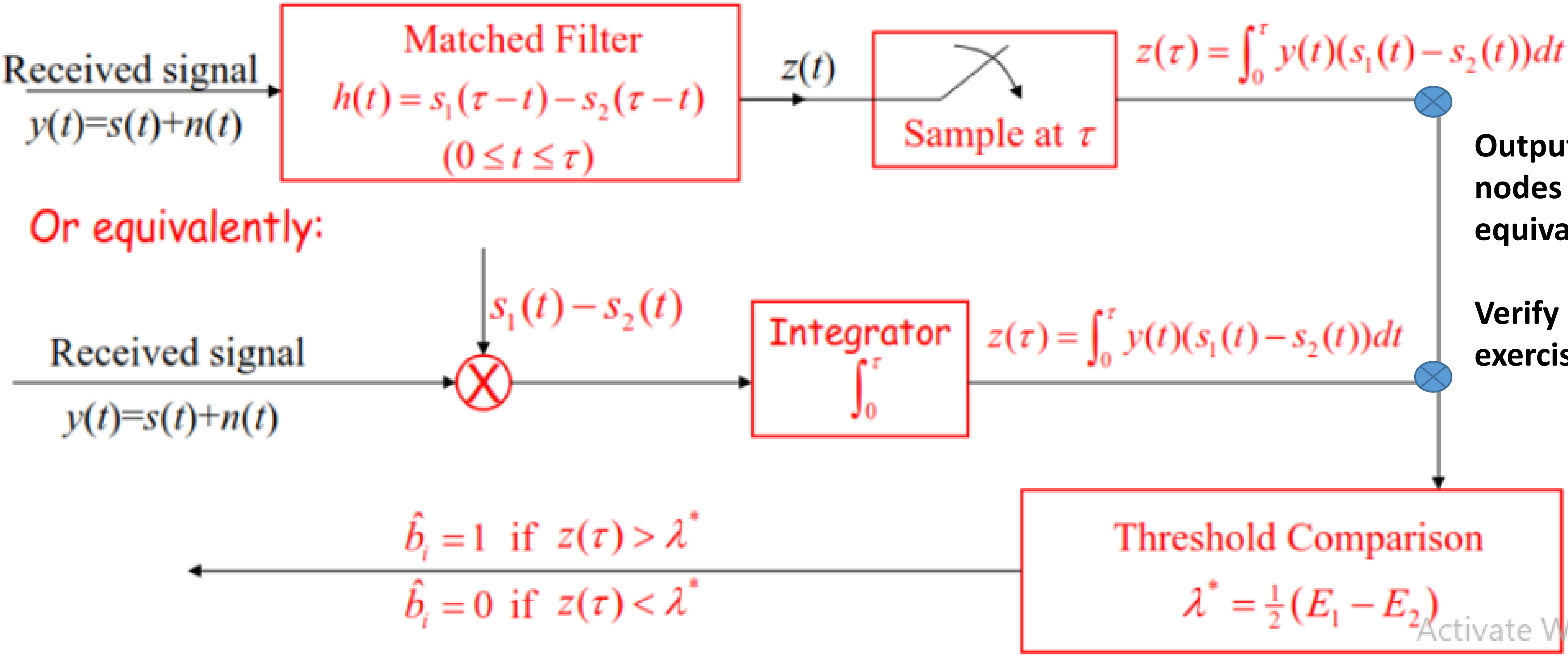
Step 1: Subtract s2 from s1.

Step 2: Rotate around the y-axis

Step 3: Translate to the right by T

Equivalent Implementations of the Optimum Receiver

The structure of the optimum receiver is depicted in the figure below. Note that the receiver can be implemented in terms of the matched filter and, equivalently, in terms of a correlator (a multiplier followed by an integrator).



Outputs at both nodes are equivalent.

Verify as an exercise

Example: Antipodal Binary Transmission

Let us consider a digital binary communication system where bits 1 and 0 are represented by the signals $s_1(t)$ and $-s_1(t)$, respectively.

For this case, $E_1 = E_2 = E = \int_0^\tau (s_1(t))^2 dt$

Therefore, the threshold is $\boldsymbol{\lambda^* = (E_1 - E_2) = 0}$

The probability of error is:

$$P_b^* = Q\left(\sqrt{\frac{\int_0^\tau (s_1(t) - s_2(t))^2 dt}{2N_0}}\right) \qquad P_b^* = Q\left(\sqrt{\frac{4\int_0^\tau (s_1(t))^2 dt}{2N_0}}\right)$$

$$P_b^* = Q\left(\sqrt{\frac{\int_0^\tau (s_1(t) + s_1(t))^2 dt}{2N_0}}\right) \qquad \boldsymbol{P_b^* = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

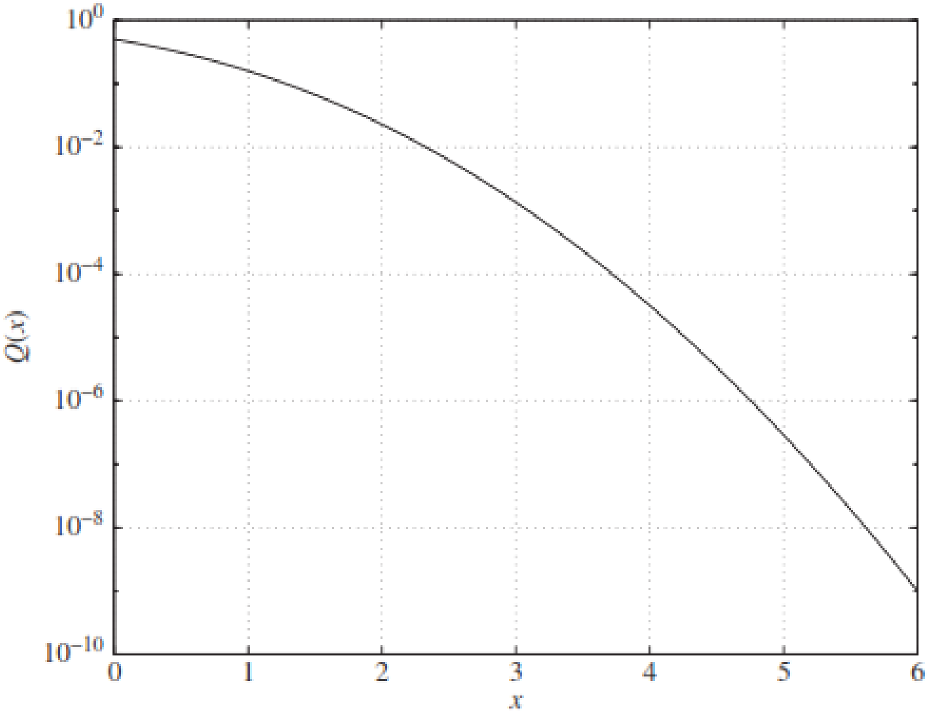
Probability of error decreases as the signal to noise ratio increases

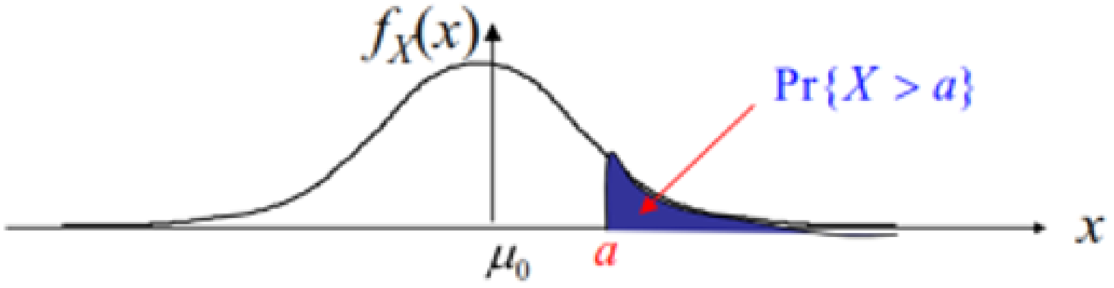
The Q-Function

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

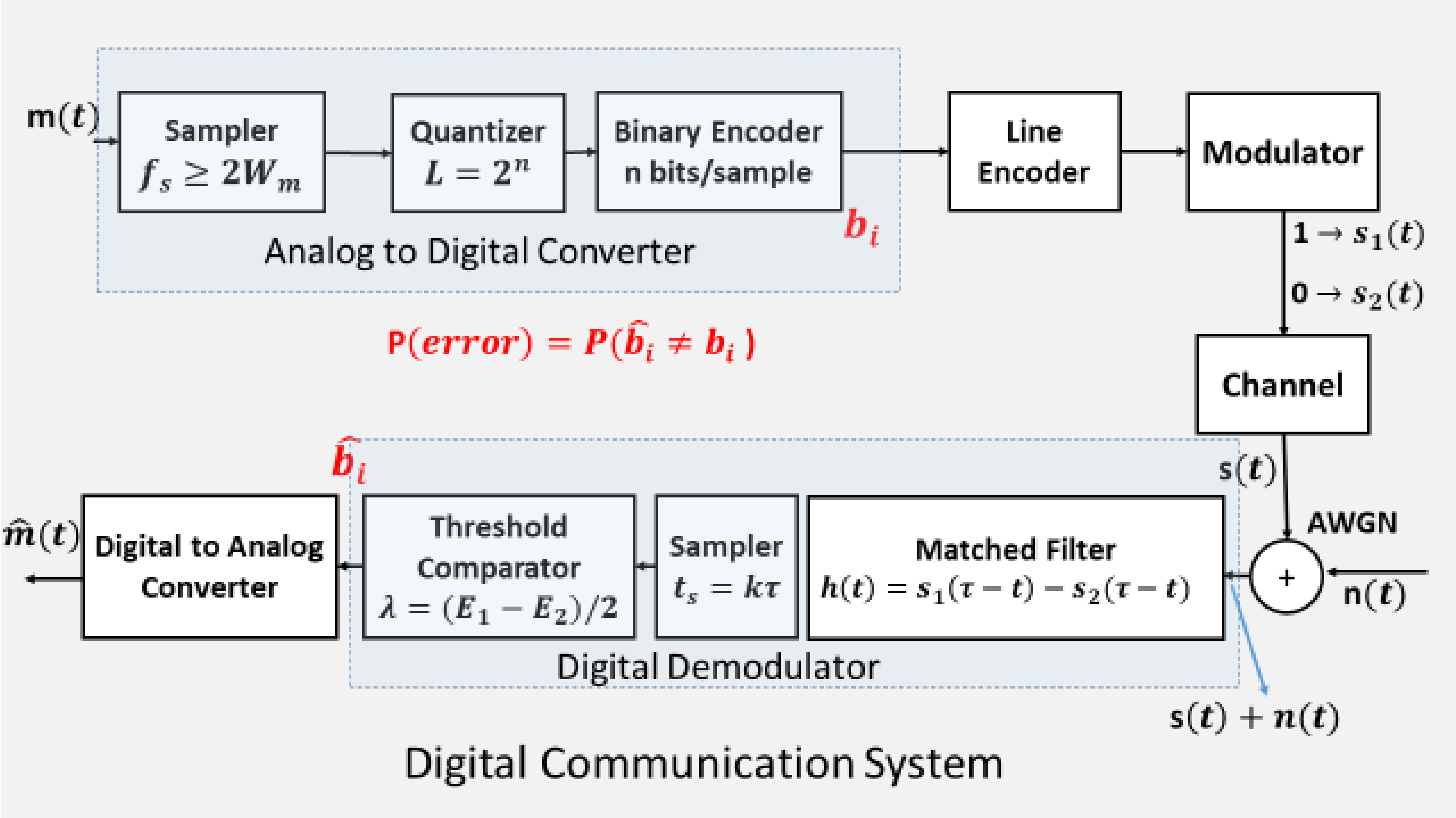
- $Q(\alpha)$ is a decreasing function of α.
- For $X \sim \mathcal{N}(\mu_0, \sigma_0^2)$,

$$\Pr\{X > a\} = \int_a^{\infty} f_X(x)dx = \int_a^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right\} dx = Q\left(\frac{a-\mu_0}{\sigma_0}\right)$$





Baseband Data Transmission (Low-pass Channel)



Baseband Data Transmission

Binary data transmission by means of two voltage levels is referred to as baseband signaling. Manchester encoding, for example, is used in the Ethernet local area network as the signaling scheme. Here, we consider polar non-return to zero baseband transmission scheme, in terms of probability of error, optimum receiver structure, power spectral density and bandwidth.

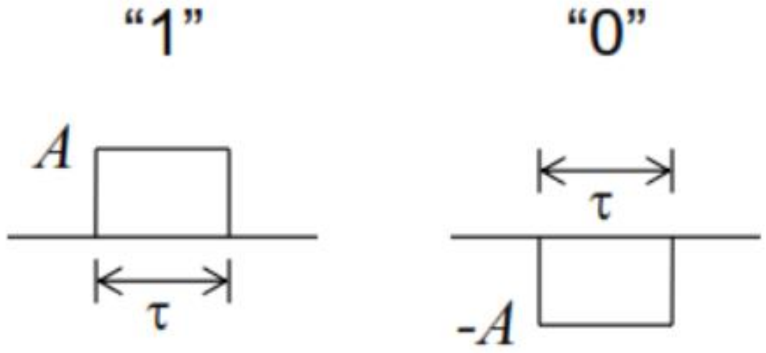
Polar nonreturn to zero (also known as binary pulse amplitude modulation)

Signal Representation

The baseband signals representing digits 1 and 0 are:

$$s_1(t) = A\,, \qquad 0 \le t \le \tau$$

$$s_0(t) = -A\,, \qquad 0 \le t \le \tau$$

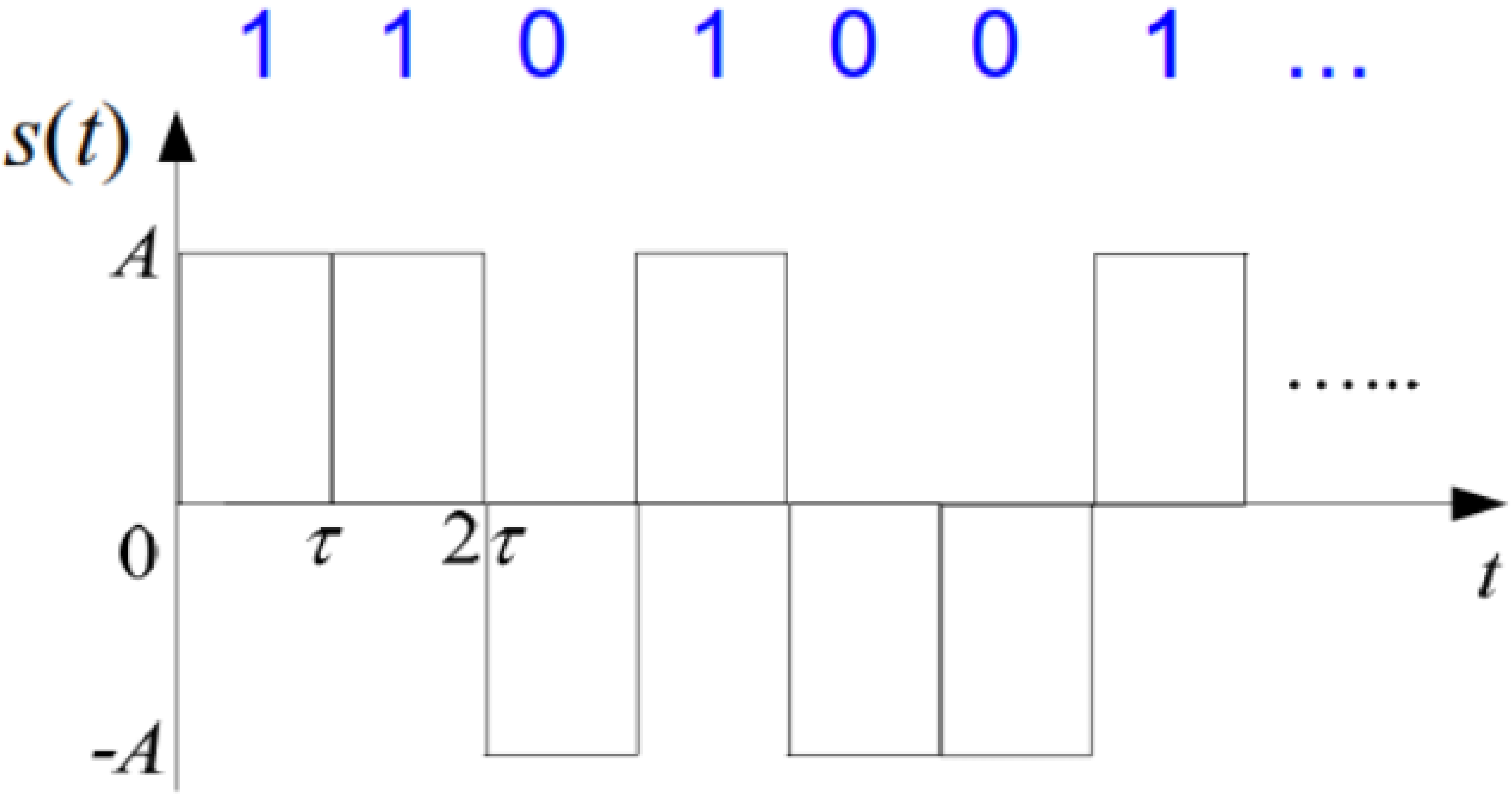


$s_1(t) = A$, $0 \le t \le \tau$ $\qquad$ $s_2(t) = -A$, $0 \le t \le \tau$

where, τ is the symbol duration and $R_b = 1/\tau$ is the data rate in bits/sec.

Baseband Data Transmission

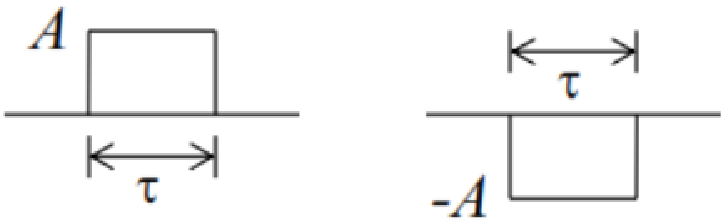
- **Generation: Convert data into polar non-return to zero format**

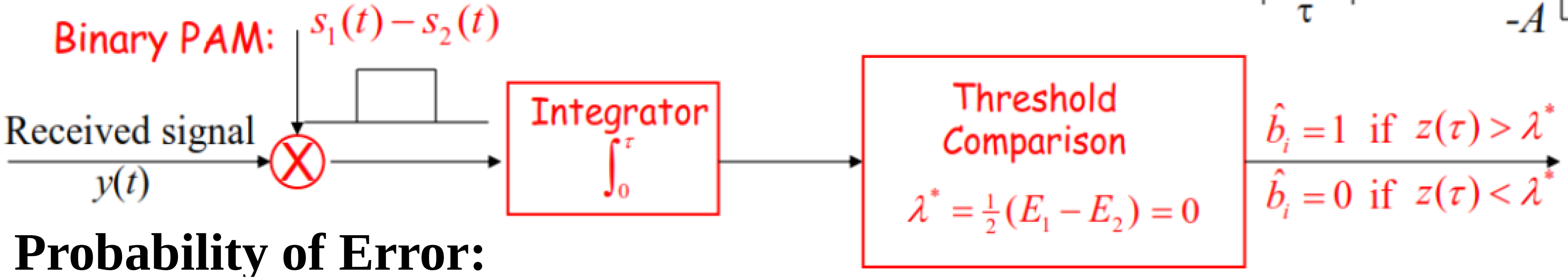


Baseband Data Transmission

Optimum Receiver

The optimum receiver is, of course, the matched filter, also implemented as a correlator, as shown in this figure.





Receiver Implemented as a Correlator

Probability of Error:

$$P_b^* = Q\left(\sqrt{\frac{\int_0^{\tau}(s_1(t) - s_2(t))^2 dt}{2N_0}}\right)$$

Note that: $E_1 = E_2 = \int_0^{\tau} A^2 dt = A^2\tau \Rightarrow \lambda^* = (E_1 - E_2) = 0$

Average Energy per bit: $E_b = \frac{1}{2}(E_1 + E_2) = A^2\tau$

Optimal BER:

$$P_b^* = Q\left(\sqrt{\frac{2A^2\tau}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

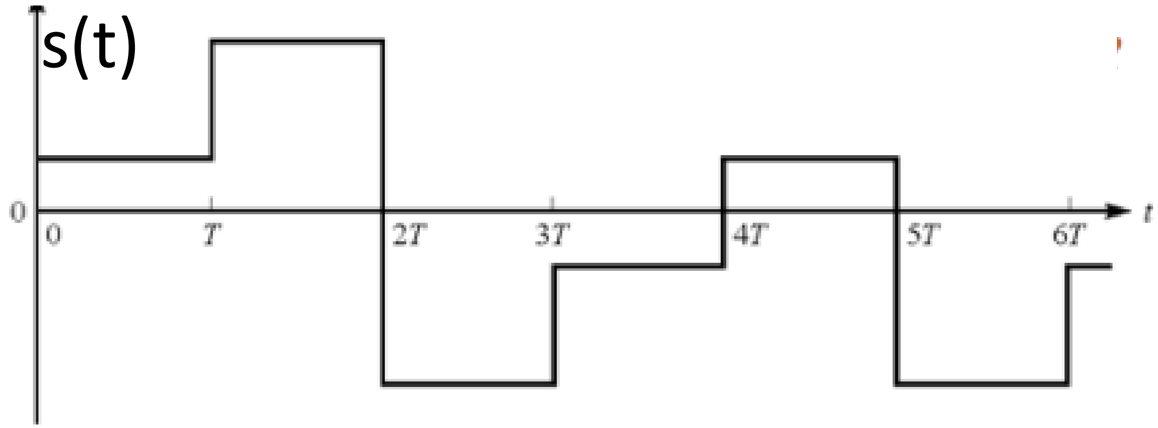
General Result on the Power Spectral Density of a digital M-ary baseband signal

The time-domain representation of a digital M-ary baseband signal is

$$s(t) = \sum_{n=-\infty}^{\infty} Z_n \cdot v(t - n\tau)$$

where Z_n is a discrete random variable with $\Pr\{Z_n = a_i\} = 1/M,\ i = 1, \ldots, M,$

$v(t)$ is a unit-baseband signal, and symbols in different time slots are assumed independent. Under these assumptions, the power spectral density of $s(t)$ is given by



$$G_s(f) = \frac{1}{\tau}|V(f)|^2 \cdot \left(\sigma_Z^2 + \frac{\mu_Z^2}{\tau} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{\tau} \right) \right)$$

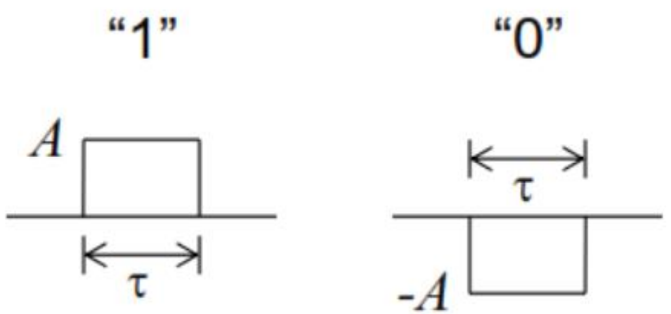
Power Spectral Density of the Polar Non-return to Zero baseband signal

The general result stated above for the M-ary baseand signal can be specialized to the polar nonreturn to zero transmission as follows

- The signal amplitude assumes two equally likely values. i.e., $P\{Z_n = \pm 1\} = 1/2$
- The basic unit pulse is $v(t) = \begin{cases} A, & 0 \le t \le \tau \\ 0, & otherwise \end{cases}$
- The Fourier transform of the basic unit pulse is $V(f) = A\tau sinc(f\tau)$
- The mean and variance of Z are: $\mu_Z = 0$, $\sigma_Z^2 = 1$

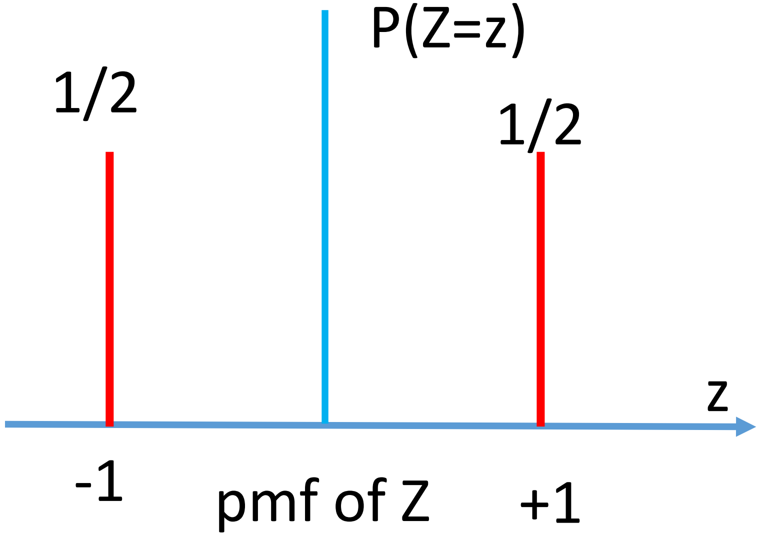
$$E(Z) = \sum_{all\ z_i} z_i P(Z = z_i)$$

$$Var(Z) = \sigma_Z^2 = \sum_{all\ z_i} (z_i - E(Z))^2 P(Z = z_i)$$



$s_1(t) = A$, $0 \le t \le \tau$

$s_2(t) = -A$, $0 \le t \le \tau$



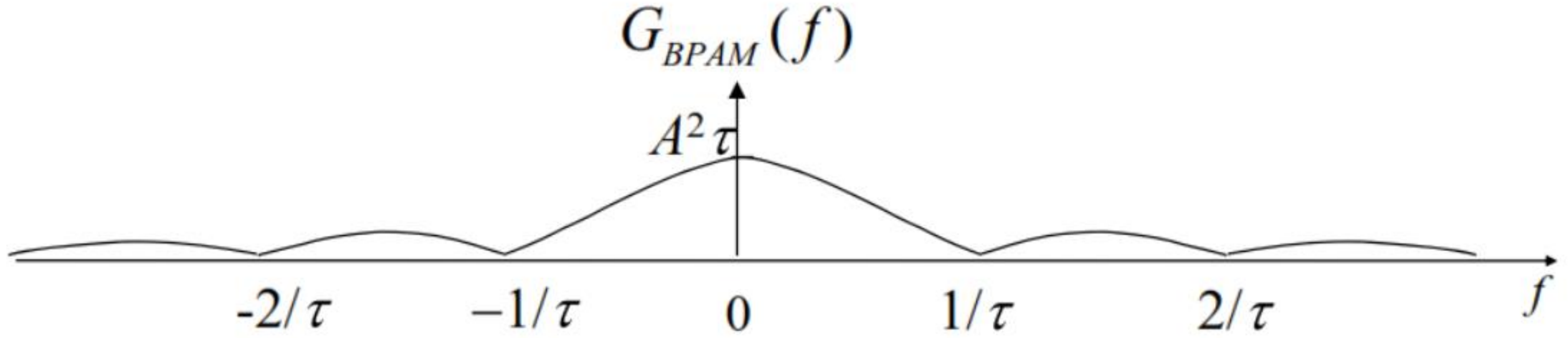
Power Spectral Density of the Polar Non-return to Zero baseband signal

- The general power spectral density formula

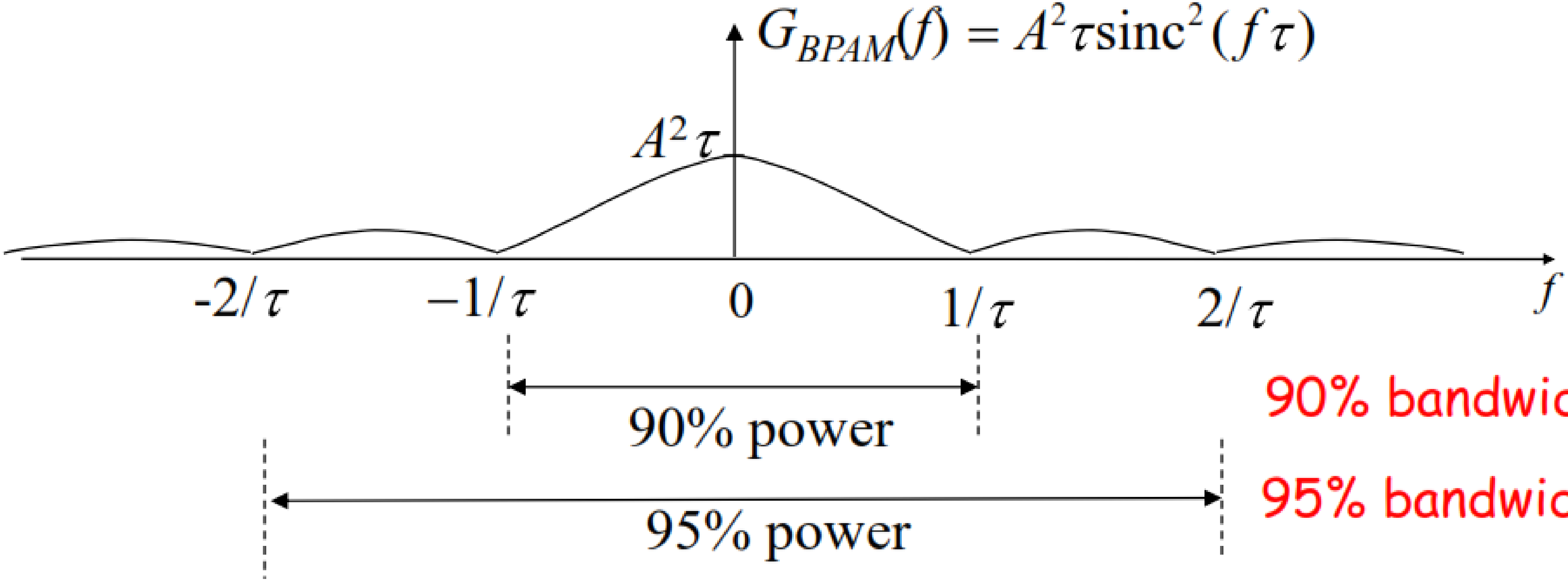
$$G_s(f) = \frac{1}{\tau}|V(f)|^2 \cdot \left(\sigma_Z^2 + \frac{\mu_Z^2}{\tau}\sum_{m=-\infty}^{\infty}\delta\left(f - \frac{m}{\tau}\right)\right)$$

- Substituting: E(Z)=0, Var(Z)=1, and the Fourier transform of the rectangular pulse v(t), we get:

$$G_{BPAM}(f) = A^2\tau\text{sinc}^2(f\tau)$$



Bandwidth of the Polar Non-return to Zero baseband signal



90% bandwidth: $1/\tau$

95% bandwidth: $2/\tau$

The 90% power bandwidth $= \frac{1}{\tau} = R_b$ (data rate)

The 95% power bandwidth $= \frac{2}{\tau} = 2R_b$ (twice the data rate)

Binary Phase Shift Keying (BPSK)

Binary Digital Bandpass Modulation

Here, the baseband data modulates a high frequency carrier to produce a modulated signal, whose spectrum is centered on the carrier frequency. We will consider four types of bandpass transmission schemes; Amplitude Shift Keying (ASK), Phase Shift Keying (PSK), Frequency Shift Keying (FSK), and Quadri-phase Shift Keying (QPSK). For each type, we consider the generation, optimum receiver, probability of error, power spectral density, and bandwidth.

Binary Phase Shift Keying: Signal Representation

Signal Representation:

Send: $s_1(t) = A\cos(2\pi f_c t)$ if the information bit is "1";

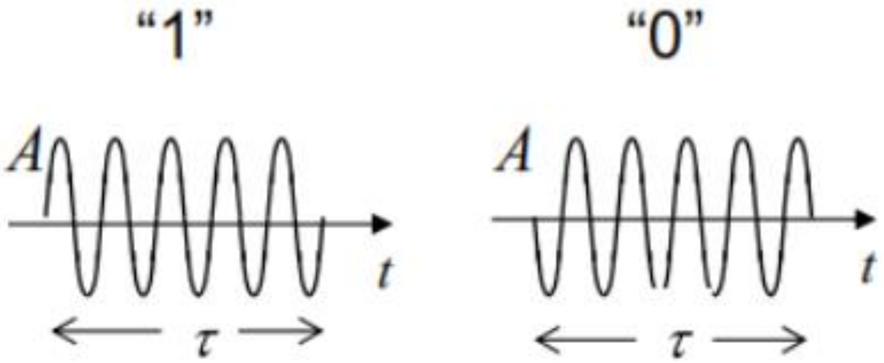
Send: $s_2(t) = A\cos(2\pi f_c t + \pi)$

$s_2(t) = -A\cos(2\pi f_c t)$ if the information bit is "0";

$\tau = nT_c$

τ: is the time allocated to transmit the binary digit.

T_c=1/f_c is the carrier period



$s_1(t) = A\cos(2\pi f_c t)$ $s_2(t) = A\cos(2\pi f_c t + \pi)$

$0 \leq t \leq \tau$ $0 \leq t \leq \tau$

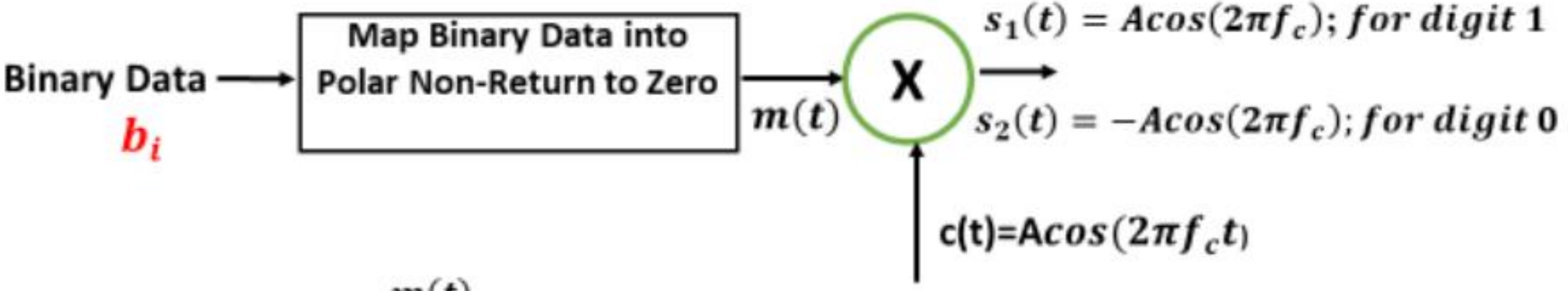
(τ is an integer number of $1/f_c$)

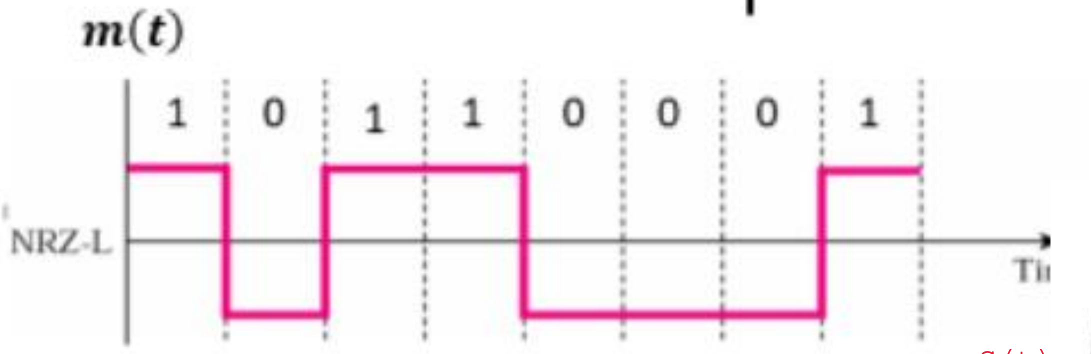
$s_2(t) = s_1(t + \pi) = -s_1(t)$

$\tau = nT_c$

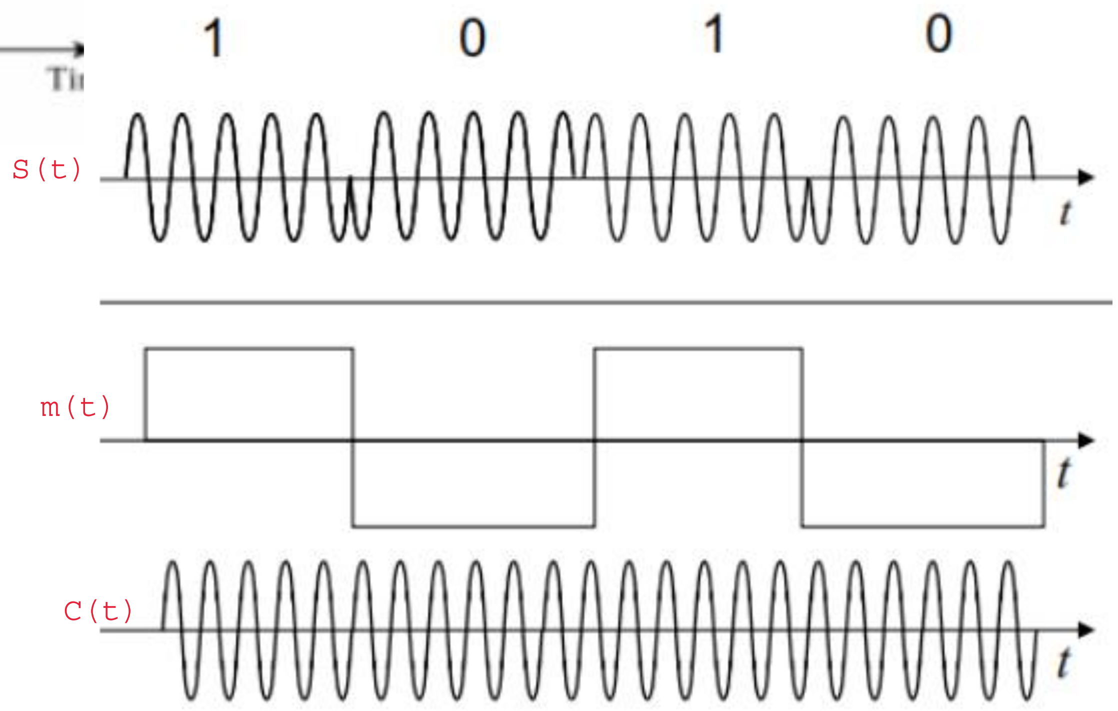
In this figure n=5

Binary Phase Shift Keying: Generation

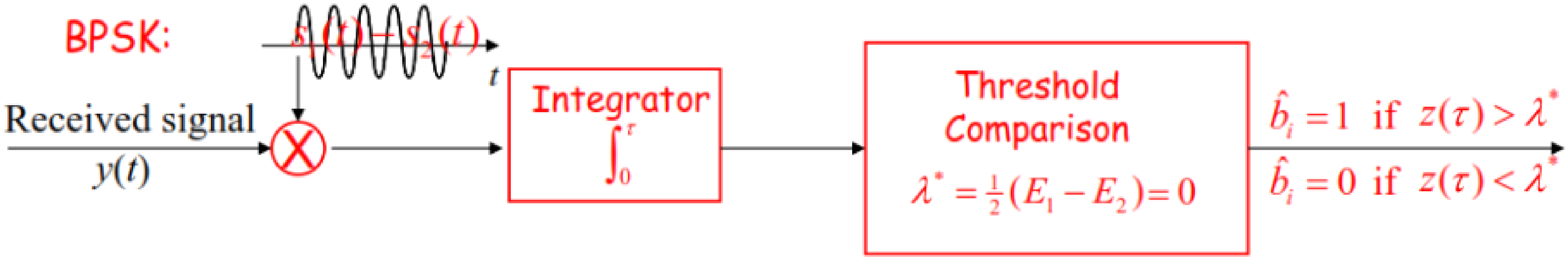




Polar Non-return to Zero



Binary Phase Shift Keying: The Optimum Receiver



Optimum Receiver Implemented as a Correlator Followed by a Threshold Detector

$$E = \int_0^{\tau} (s(t))^2 dt$$

With $\tau = nT_c$

$$E = A^2\tau/2$$

Verify this result

$$s_1(t) = A\cos(2\pi f_c t)$$

$$s_2(t) = -A\cos(2\pi f_c t)$$

Probability of Error:

$$P_b^* = Q\left(\sqrt{\frac{\int_0^{\tau}(s_1(t) - s_2(t))^2 dt}{2N_0}}\right)$$

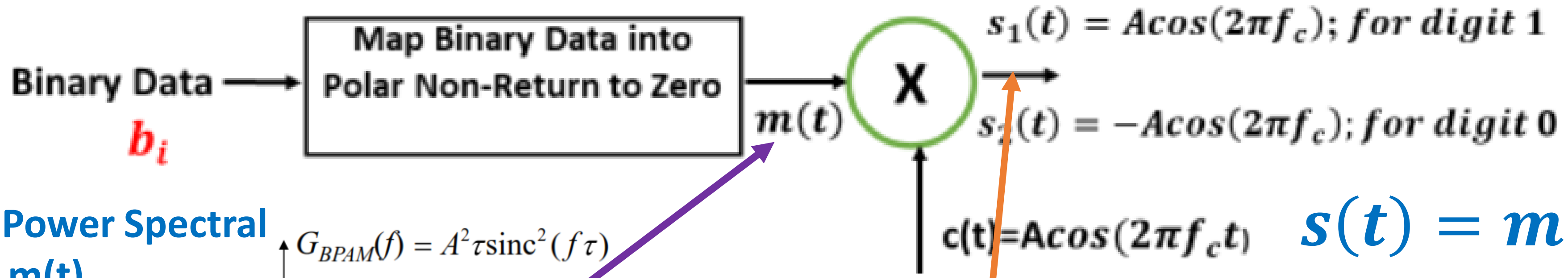
Energy of $s_i(t)$: $E_1 = E_2 = \frac{1}{2}A^2\tau$

Average Energy per bit: $E_b = \frac{1}{2}(E_1 + E_2) = \frac{1}{2}A^2\tau$

Optimal BER:

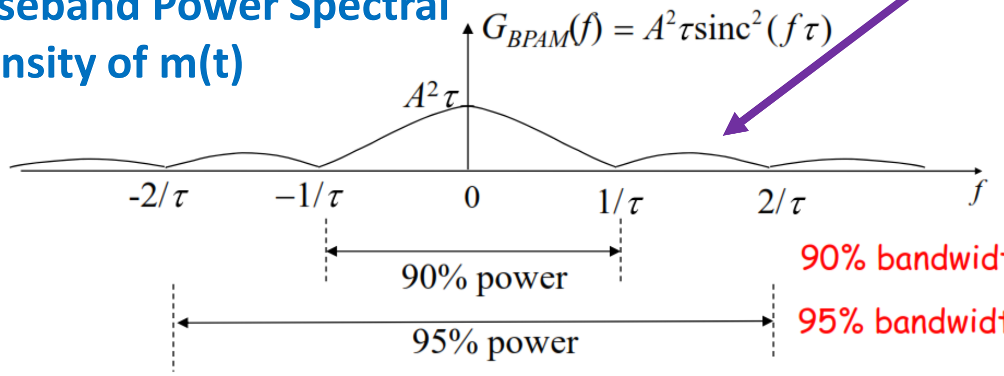
$$P_b^* = Q\left(\sqrt{\frac{A^2\tau}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Binary Phase Shift Keying: Power Spectral Density and Bandwidth



$$s(t) = m(t)\cos(2\pi f_c t)$$

Baseband Power Spectral Density of m(t)



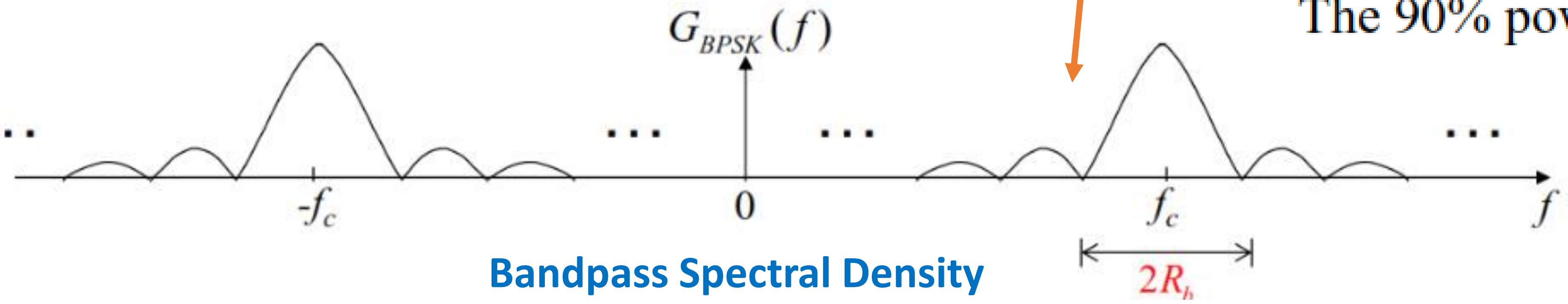
90% bandwidth: $1/\tau$

95% bandwidth: $2/\tau$

Bandwidth of BPSK s(t)

Power Spectral Density

$$G_{BPSK}(f) = \tfrac{1}{4}[G_{BPAM}(f - f_c) + G_{BPAM}(f + f_c)]$$



The 90% power bandwidth $= \frac{2}{\tau} = 2R_b$

(twice the data rate); Same as that of BASK

Bandpass Spectral Density

Extra Material on the Power Spectral Density

The Wiener –Khintchine Thorem:

The power spectral density $G_X(f)$ and the autocorrelation function $R_X(\tau)$ of a stationary random process $X(t)$ form a Fourier transform pairs:

$G_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$ (Fourier Transform)

$R_X(\tau) = \int_{-\infty}^{\infty} G_X(f) e^{j2\pi f\tau} df$ (Inverse Fourier Transform)

Example: Mixing of a random process with a sinusoidal signal.

- A random process $X(t)$ with an autocorrelation function $R_X(\tau)$ and a power spectral density $G_X(f)$ is mixed with a sinusoidal function $\cos(2\pi f_c t + \theta)$; θ is a r.v uniformly distribution over $(0, 2\pi)$ to form a new process

$$Y(t) = X(t)\cos(2\pi f_c t + \theta).$$ Find $R_Y(\tau)$ and $G_Y(f)$

- **Solution**: We first find $R_Y(\tau)$
- $R_Y(\tau) = E\{\, Y(t)Y(t+\tau)\}$
- $= E\{X(t)\cos(2\pi f_c t + \theta) \cdot X(t+\tau)\cos(2\pi f_c t + 2\pi f_c \tau + \theta)\}$

When $X(t)$ and θ are independent, then

- $= E\{X(t)\,X(t+T)\}E\{\cos(2\pi f_c t + \theta) \cdot \cos(2\pi f_c t + 2\pi f_c \tau + \theta)\}$
- $= R_X(\tau)E\{\frac{\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) + \cos 2\pi f_c \tau}{2}\}$
- $\boldsymbol{R_Y(\tau) = \frac{R_X(\tau)}{2} \cdot \cos 2\pi f_c \tau}$;
- The power spectral density is
- $\boldsymbol{S_Y(f) = \frac{1}{4}\{G_X(f - f_c) + G_X(f + f_c)\}}$
- Which is quite similar to the modulation property of the Fourier transform.

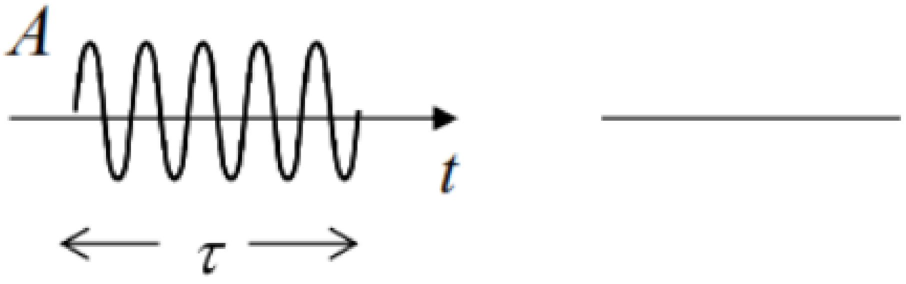
Binary Amplitude Shift Keying (BASK): Signal Representation

Send: $s_1(t) = A\cos(2\pi f_c t)$, if the information bit is "1" $\Rightarrow E_1 = \frac{A^2\tau}{2}$

Send: $s_2(t) = 0$, if the information bit is "0"; $\Rightarrow E_2 = 0$

The average energy per bit $E_b = \frac{1}{2}(E_1 + E_2) = \frac{A^2\tau}{4}$

"1" "0"



$s_1(t) = A\cos(2\pi f_c t)$ $s_2(t) = 0$

$0 \le t \le \tau$

$\tau = nT_c$

τ: is the time allocated to transmit the binary digit.

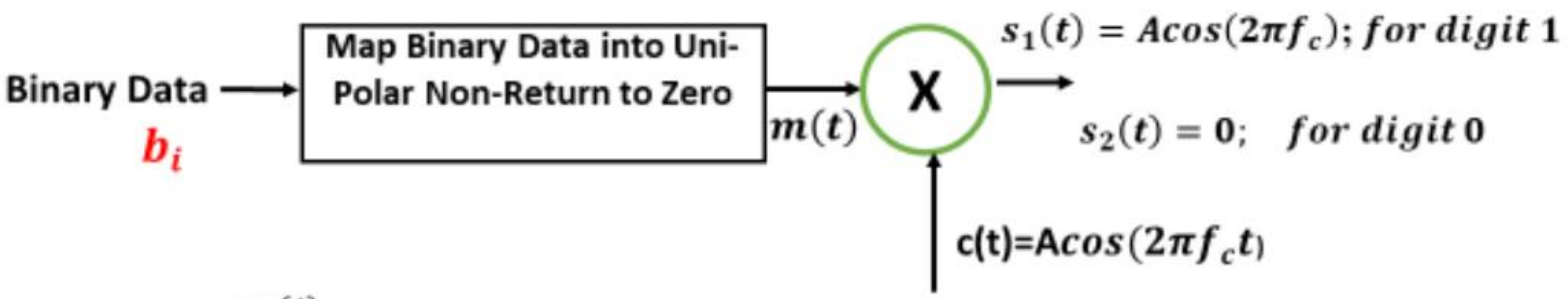
$T_c = 1/f_c$ is the carrier period

$R_b = \frac{1}{\tau}: Data\ rate$ bits/sec

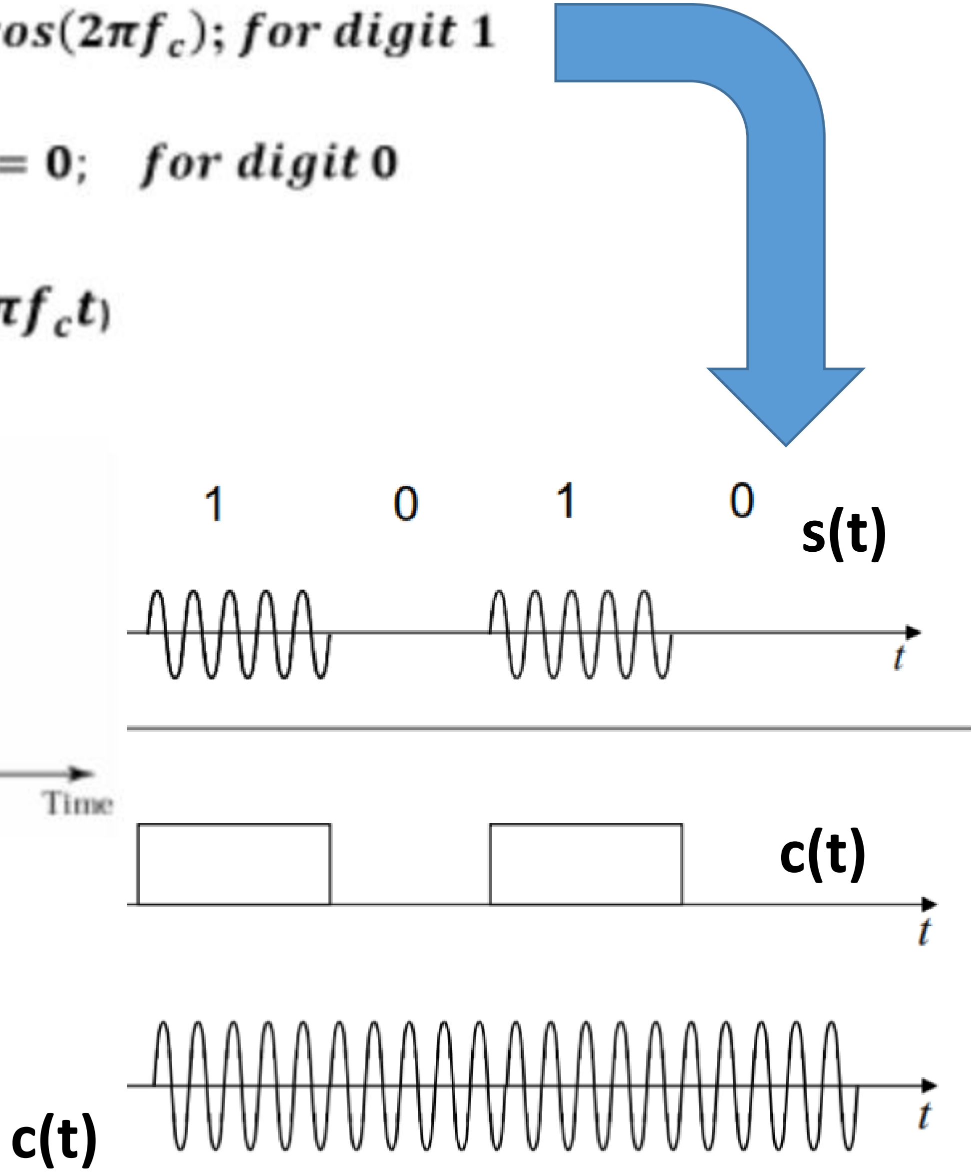
$\tau = nT_c$

In this figure n=5

Binary Amplitude Shift Keying : Generation



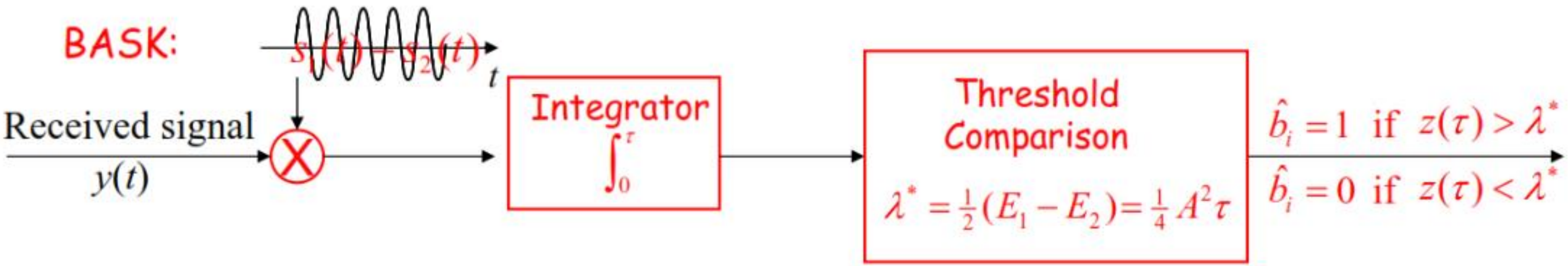
m(t): Unipolar non-return to zero



$$s(t) = m(t)c(t)$$

NOTE: BASK is also called ON-OFF Keying

Binary Amplitude Shift Keying : The Optimum Receiver



Optimum Receiver Implemented as a Correlator Followed by a Threshold Detector

Probability of Error:

$$P_b^* = Q\left(\sqrt{\frac{\int_0^\tau (s_1(t) - s_2(t))^2 dt}{2N_0}}\right)$$

$$E_1 = \int_0^\tau (s_1(t))^2 dt$$

With $\tau = nT_c$

$$E_1 = A^2\tau/2$$

$$s_1(t) = A\cos(2\pi f_c t)$$

$$s_2(t) = 0,$$

$$E_1 = \frac{A^2\tau}{2}$$

$$E_2 = 0$$

$$E_b = \frac{1}{2}(E_1 + E_2) = \frac{A^2\tau}{4}$$

Optimal BER:

$$P_b^* = Q\left(\sqrt{\frac{A^2\tau}{4N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Binary Amplitude Shift Keying: Power Spectral Density

Let $\mathrm{m}(t)$ be the unipolar NRZ signal with autocorrelation function $R_m(\tau)$ and power spectral density $G_m(f)$.

- You can easily verify that the unipolar non-return to zero signal $m(t)$ is related to the polar non-return signal $m'(t)$ (used in the generation of the BPSK) by:

- $m(t) = \frac{1}{2}(1 + m'(t)$

  The autocorrelation function of m(t) is

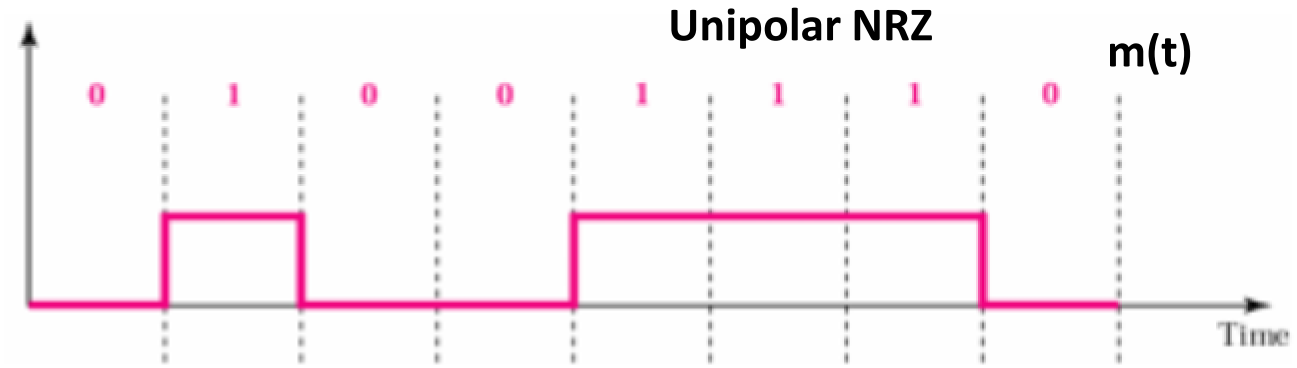
- $R_Y(\tau) = E\{\, m(t)m(t+\tau)\} =$

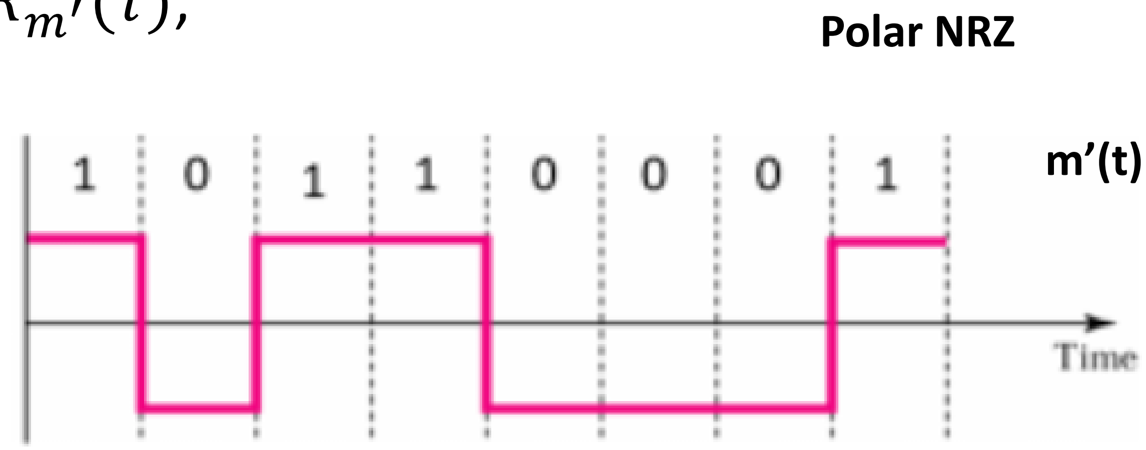
$= E\left\{\frac{1}{2}(1 + m'(t)\frac{1}{2}(1 + m'(t+\tau)\right\} = \frac{1}{4} + \frac{1}{4}R_{m'}(\tau);$

  Note that for the polar-NRZ $\boldsymbol{E\{m'(t)\} = 0}$

- The power spectral density of m(t) is:

- $G_m(f) = \frac{1}{4}\delta(f) + \frac{1}{4}G_{m'}(f)$





Binary Amplitude Shift Keying: Power Spectral Density

The Wiener –Khintchine Thorem: The power spectral density $G_X(f)$ and the autocorrelation function $R_X(\tau)$ of a stationary random process $X(t)$ form a Fourier transform pairs:

- $G_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$ (Fourier Transform)
- $R_X(\tau) = \int_{-\infty}^{\infty} G_X(f) e^{j2\pi f\tau} df$ (Inverse Fourier Transform)

- The power spectral density of m(t), the unipolar NRZ is:
- $\boldsymbol{G_m(f) = \frac{1}{4}\delta(f) + \frac{1}{4}G_{m'}(f)}$
- In the previous video we saw that if a random process $X(t)$ with an autocorrelation function $R_X(\tau)$ and a power spectral density $G_X(f)$ is mixed with a sinusoidal function $\cos(2\pi f_c t + \theta)$; θ is a r.v uniformly distribution over $(0, 2\pi)$ to form a new process

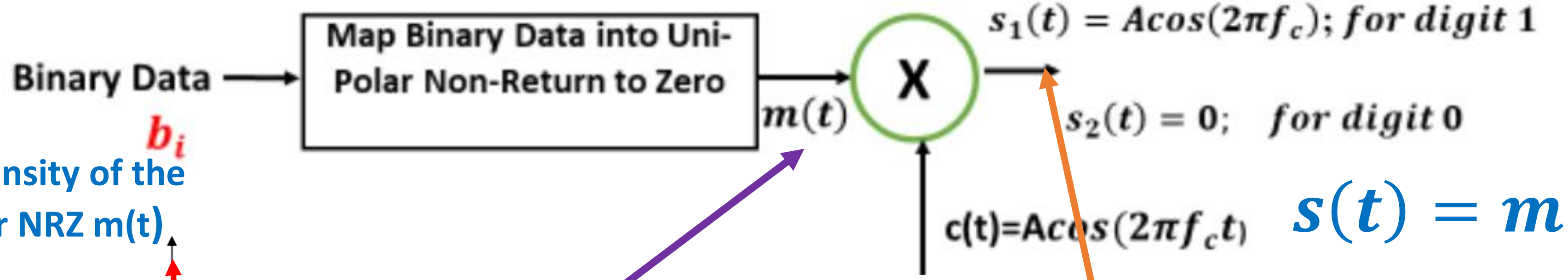
$$Y(t) = X(t)\cos(2\pi f_c t + \theta).$$

Our Problem: $\boldsymbol{s(t) = m(t)\cos(2\pi f_c t + \theta)}$

then the autocorrelation function and power spectral density of Y(t) are given by:

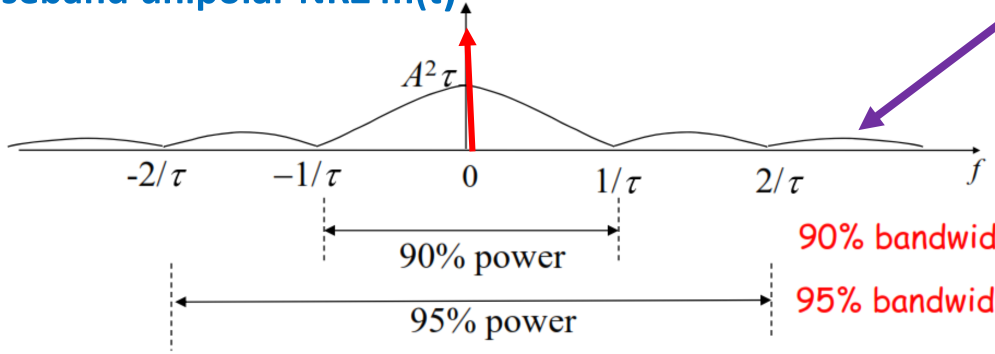
- $R_Y(\tau) = E\{Y(t)Y(t+\tau)\} = \boldsymbol{\frac{R_X(\tau)}{2} \cdot \cos 2\pi f_c \tau}$;
- $\boldsymbol{G_Y(f) = \frac{1}{4}\{G_X(f - f_c) + G_X(f + f_c)\}}$
- **Hence,** $\boldsymbol{G_{BASK}(f) = \frac{1}{4}\{G_m(f - f_c) + G_m(f + f_c)}$

Binary Amplitude Shift Keying : Power Spectral Density and Bandwidth



$$s(t) = m(t)\cos(2\pi f_c t)$$

Power Spectral Density of the baseband unipolar NRZ m(t)



90% bandwidth: $1/\tau$

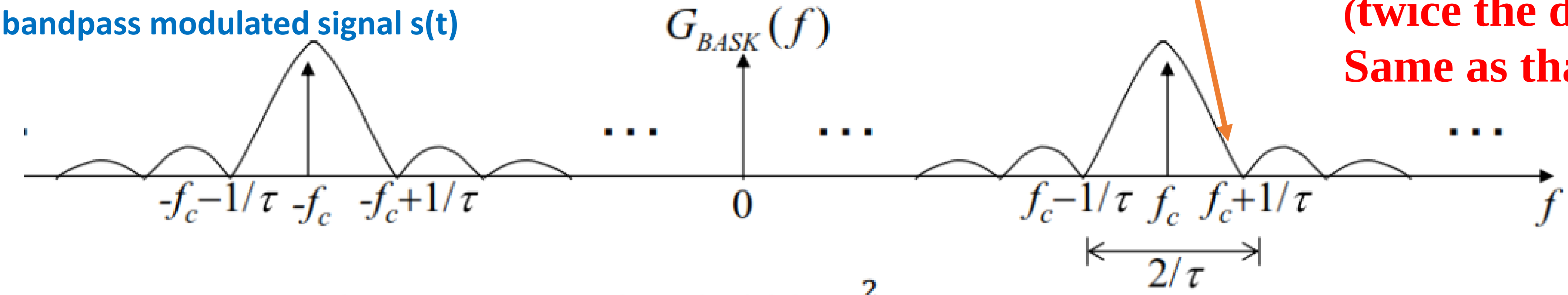
95% bandwidth: $2/\tau$

$$G_{BASK}(f) = \frac{1}{4}[G_{BOOK}(f - f_c) + G_{BOOK}(f + f_c)]$$

Bandwidth of BASK s(t)

(twice the data rate); Same as that of BPSK

Power Spectral Density of the bandpass modulated signal s(t)

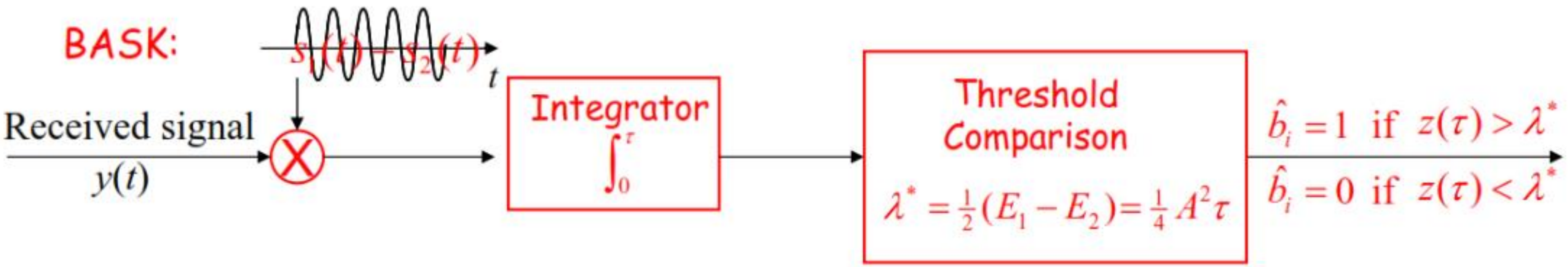


$2R_b$

The 90% power bandwidth $= \frac{2}{\tau} = 2R_b$

Non-coherent Demodulation of the Binary ASK Signal

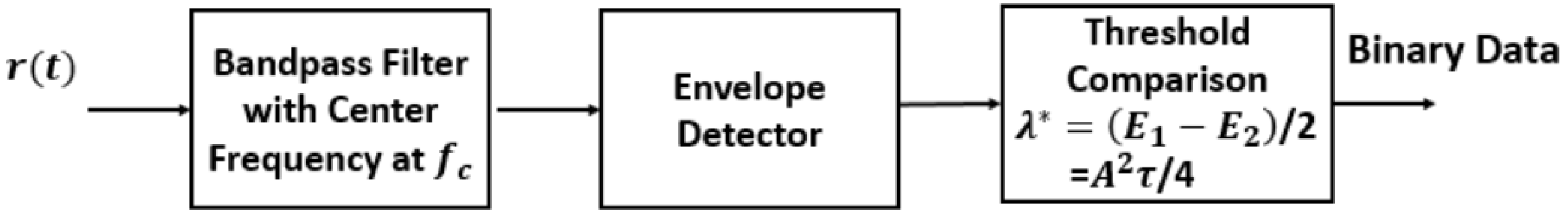
The demodulator which uses the signal difference $s_1(t) - s_2(t) = Acos(2\pi f_c t)$ is called coherent demodulator



In non-coherent demodulation, there is no need for the carrier frequency at the receiver. The basic elements of the receiver are a bandpass filter with center frequency at the carrier, an envelope detector, and a threshold comparator. The receiver is simple, however it is not optimal in terms of the probability of error. The details are shown in the following block diagram

$$\mathbf{r(t)} = Acos(2\pi f_c t) + n(t)$$

$$\mathbf{r(t)} = n(t)$$



Non-Coherent Binary ASK Demodulation

Binary Frequency Shift Keying (BFSK): Signal Representation

In binary FSK, the frequency of the carrier signal is varied to represent the binary digits and 0 by two distinct frequencies. The amplitude and frequency remain constant during each bit interval.

Signal Representation (coherent FSK)

Send: $s_1(t) = Acos(2\pi(f_c + \Delta f))t)$ if the information bit is "1";

Send: $s_2(t) = Acos(2\pi(f_c - \Delta f))t)$ if the information bit is "0";

Δf is an offset frequency (from the unmodulated carrier f_c) chosen so that $s_1(t)$ and $s_2(t)$ are orthogonal, i.e.,

$$\int_0^{\tau} s_1(t)s_2(t)dt = 0$$

$$\frac{sin(2\pi(2f_c)\tau}{2f_c} + \frac{sin(2\pi(2\Delta f)\tau}{2\Delta f} = 0$$

Binary Frequency Shift Keying (BFSK): Signal Representation

Orthogonality condition

$$\frac{sin(2\pi(2f_c)\tau}{2f_c} + \frac{sin(2\pi(2\Delta f)\tau}{2\Delta f} = 0$$

$$2f_c = \frac{n}{2\tau} = \frac{nR_b}{2}, n = 1, 2, \ldots. \quad \boldsymbol{f_c = \frac{nR_b}{4} = kR_b}$$

$$2\Delta f = \frac{mR_b}{2}, m = 1, 2, \ldots. \quad \boldsymbol{\Delta f = \frac{mR_b}{4}}$$

Note that $sin(x) = 0$ when $x = n\pi$

The minimum frequency separation $\boldsymbol{2\Delta f = R_b/2.}$

"1" "0"

$s_1(t) = A\cos(2\pi(f_c + \Delta f)t)$ $\quad$ $s_2(t) = A\cos(2\pi(f_c - \Delta f)t)$

$0 \le t \le \tau$ $\quad$ $0 \le t \le \tau$

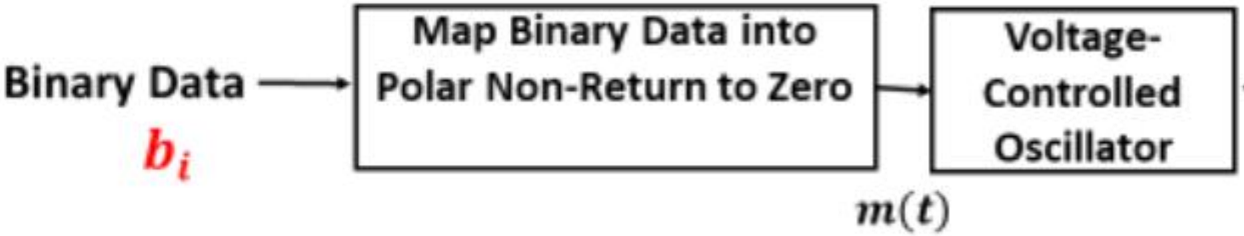
(τ is an integer number of $1/(f_c \pm \Delta f)$)

τ: is the time allocated to transmit the binary digit.

T_c=$1/f_c$ is the carrier period

$R_b = \frac{1}{\tau}$: *Data rate* bits/sec

Binary FSK : Generation using the Single Oscillator Method

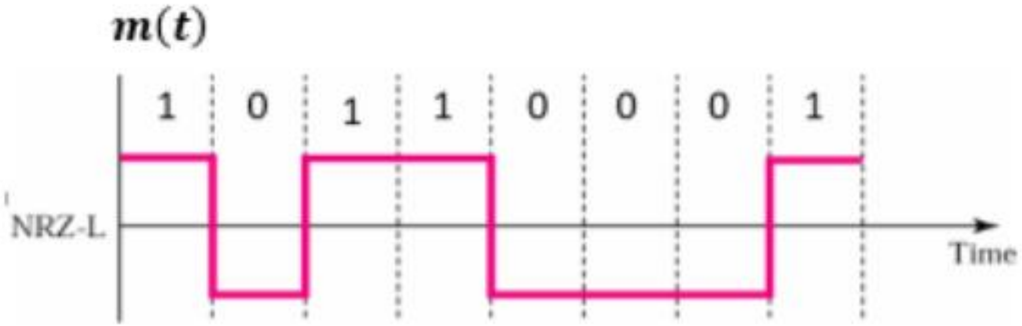


$$f_1(t) = f_c + \Delta f; \text{for digit } 1$$

$$f_2(t) = f_c - \Delta f; \text{for digit } 0$$

m(t): Polar non-return to zero

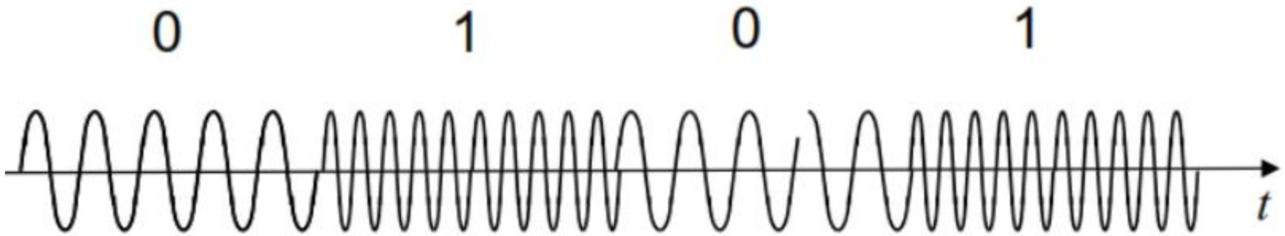
$$f_i(t) = f_c + k_f m(t); \text{for VCO}$$



Polar Non-return to Zero

$$s_1(t) = A\cos(2\pi(f_c + \Delta f))t)$$

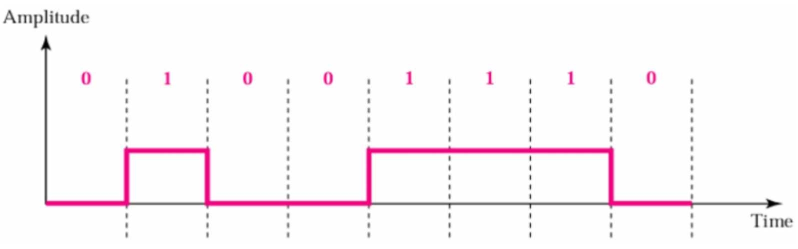
$$s_2(t) = A\cos(2\pi(f_c - \Delta f))t)$$



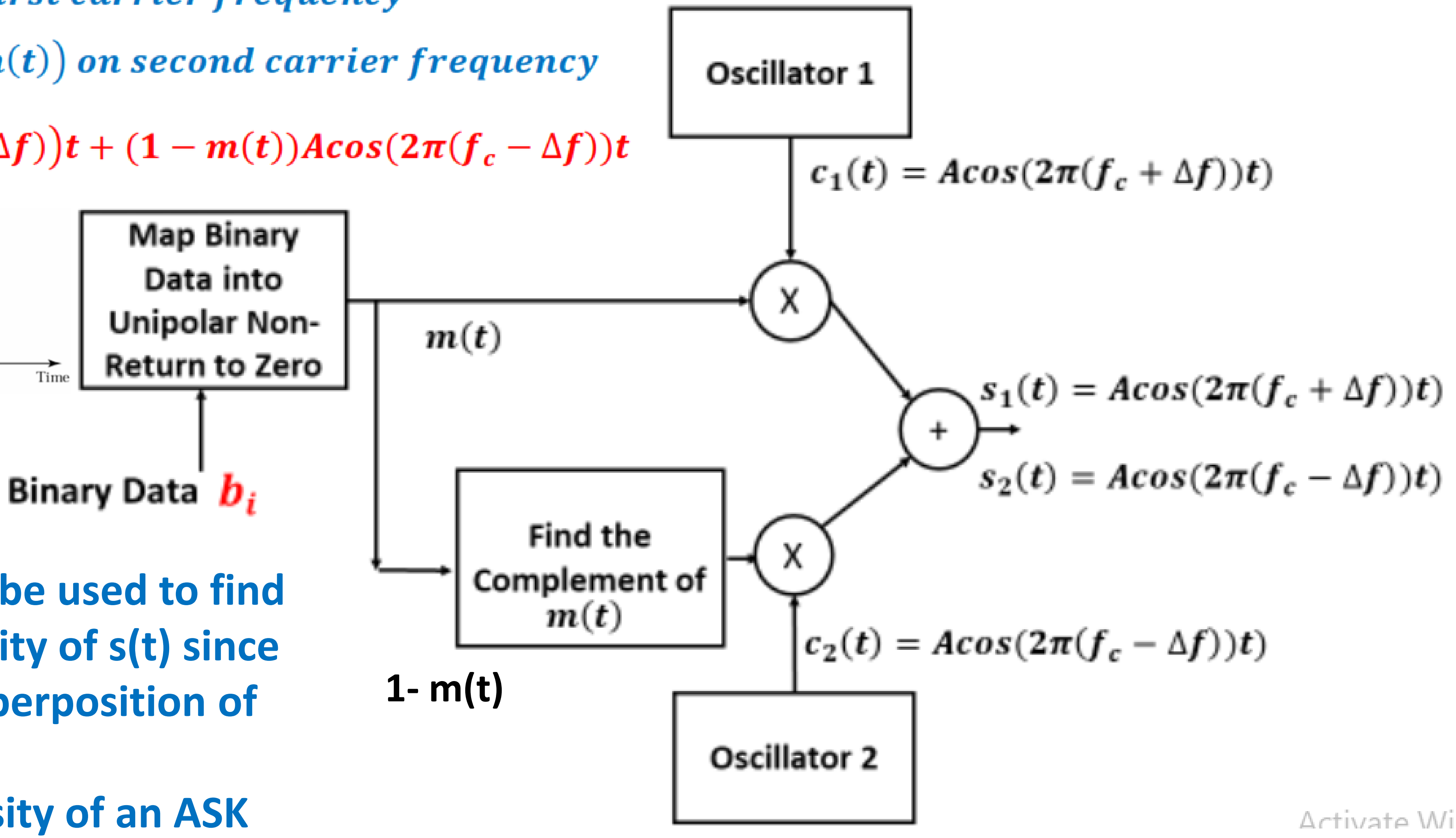
Binary FSK : Generation using the Two-oscillator Method

$$s_{BFSK}(t) = ASK\ of\ m(t)\ on\ first\ carrier\ frequency$$
$$+\ ASK\ of\ (1 - m(t))\ on\ second\ carrier\ frequency$$

$$s_{BFSK}(t) = m(t)A\cos(2\pi(f_c + \Delta f))t + (1 - m(t))A\cos(2\pi(f_c - \Delta f))t$$



Unipolar Non-Return to Zero Signal

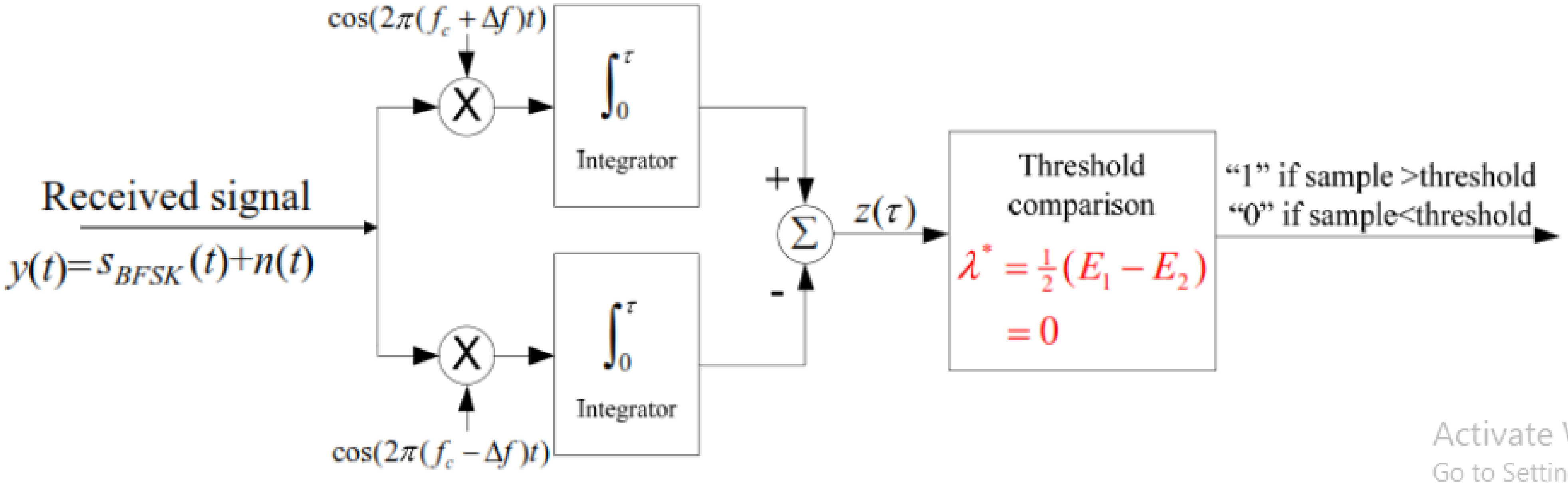


- **This representation will be used to find the power spectral density of s(t) since it is envisaged as the superposition of two ASK signals.**
- **The power spectral density of an ASK signal was derived in a previous video titled: Binary ASK**

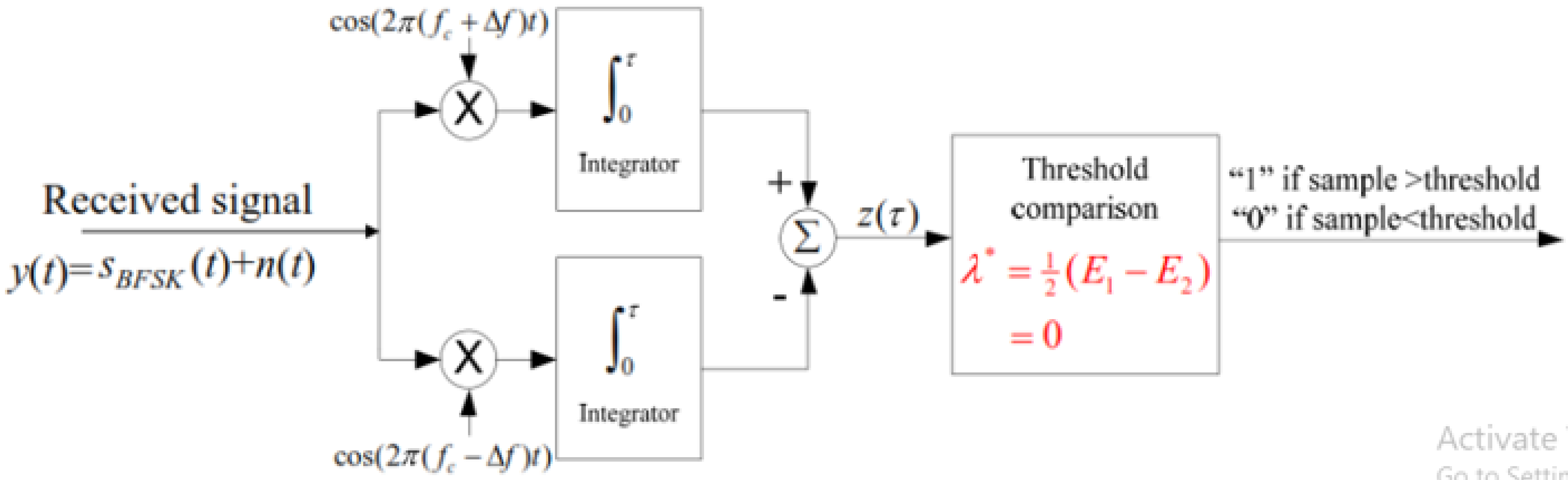
FSK: modeled as a sum of two ASK signals

Binary FSK : Coherent Demodulation

The optimum coherent receiver consists of two correlators. The operation of the receiver makes use of the orthogonality condition imposed on the signals $s_1(t)$ and $s_2(t)$. In the absence of noise, if $s_1(t)$ is received, then the ouput of the upper correlator will have a value greater than zero, while the output of the lower correlator is zero. The converse is true when $s_2(t)$ is received. In the presence of noise, the system decides 1 when $z(\tau) > 0$. That is, when the output of the upper correlator is greater than the output of the lower one. Otherwise, it decides 0.



Binary FSK : Probability of Error



Probability of Error

Energy of $s_i(t)$: $E_1 = E_2 = \frac{1}{2}A^2\tau$

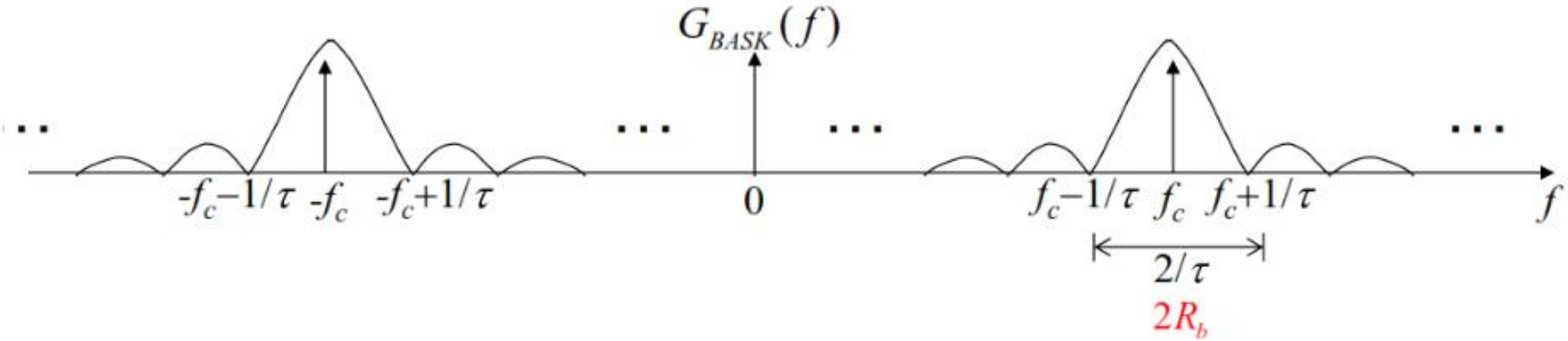
Average Energy per bit: $E_b = \frac{1}{2}(E_1 + E_2) = \frac{1}{2}A^2\tau$

When the signals are orthogonal, i.e., when $\int_0^\tau s_1(t)s_2(t)dt = 0$, the probability of error is given by

$$P_b^* = Q\left(\sqrt{\frac{A^2\tau}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Binary FSK : Power Spectral Density

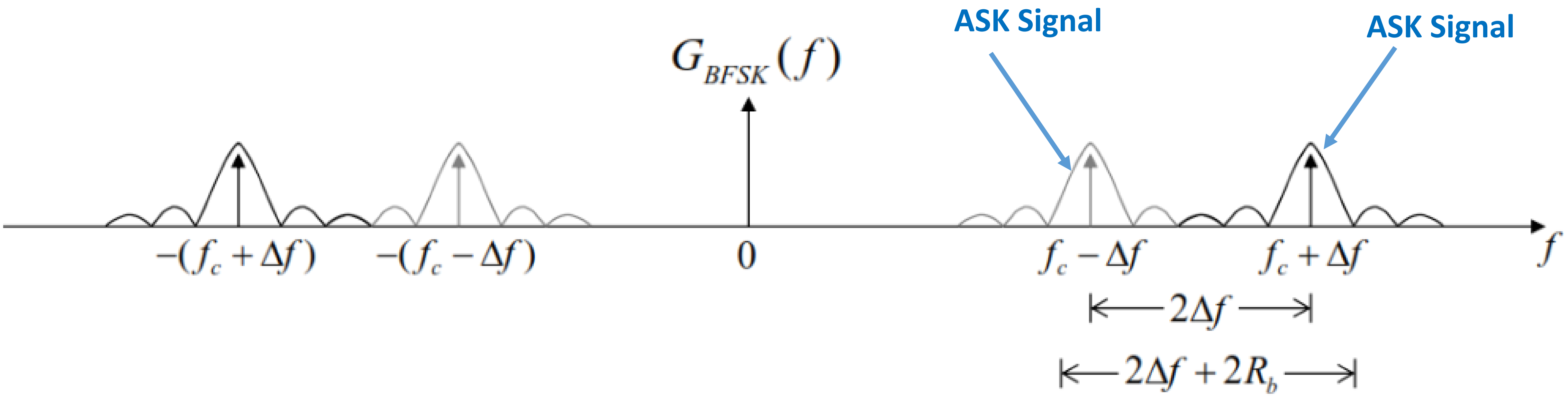
Since the FSK signal is the superposition of two ASK signals on two orthogonal frequencies, the spectrum is also the superposition of that of the ASK signals. We recall that the spectrum of the ASK signal is as shown below



$$s_{BFSK}(t) = ASK\ of\ m(t)\ on\ first\ carrier\ frequency$$
$$+\ ASK\ of\ (1-m(t))\ on\ second\ carrier\ frequency$$

$$s_{BFSK}(t) = m(t)A\cos(2\pi(f_c+\Delta f))t + (1-m(t))A\cos(2\pi(f_c-\Delta f))t$$

Binary FSK : Power Spectral Density and Bandwidth

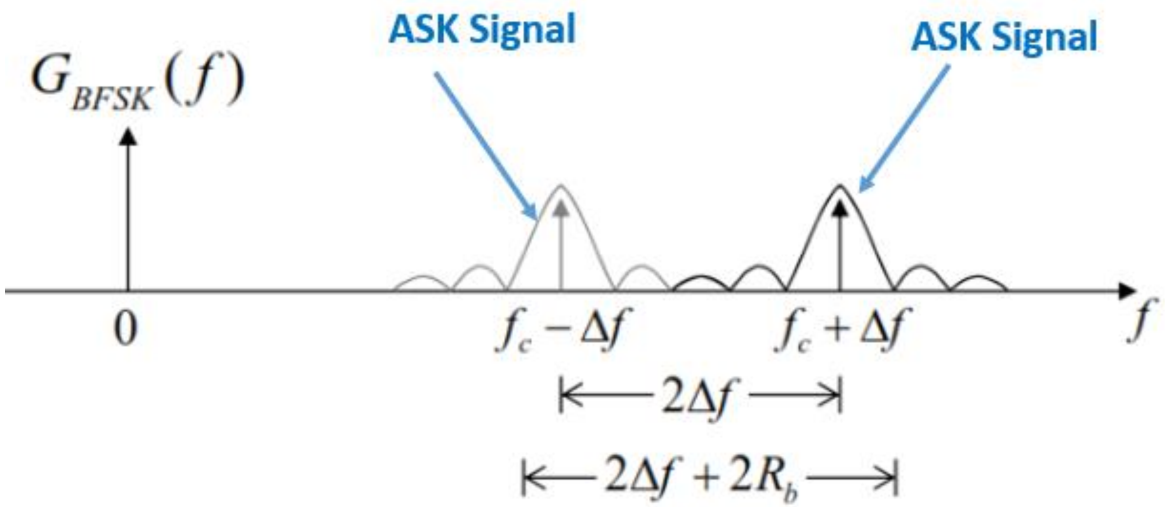


The required channel bandwidth for 90% in-band power

$$B_{h\_90\%} = 2\Delta f + 2R_b$$

$$\mathbf{B.W} = (f_1 - f_2) + 2R_b = \frac{R_b}{2} + 2R_b$$

Binary FSK : Non-coherent Demodulation



$$r(t) = A\cos(2\pi(f_c + \Delta f)t) + n(t)$$

$$r(t) = A\cos(2\pi(f_c - \Delta f)t) + n(t)$$